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THE FORMATION OF THE LESSER ANTILLES

By W. M. Davis

Harvard University

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The Islands.—The island chain of the Lesser Antilles extends in a curve about 500 miles in length between Porto Rico and Trinidad, and includes 25 larger members with an uncounted number of smaller ones. Most of the islands are of volcanic origin, now in various stages of dissection; but five of the larger islands and two small ones consist wholly of calcareous strata, and five others are composed of both volcanic rocks and calcareous strata. Vieques and St. Croix, southeast of Porto Rico, and Barbados, east of the southern part of the chain, are of other origins. The following inferences regarding the history of the islands are based on observations made during a voyage in October and November, 1923, and on a study of all available charts and published articles. A fuller statement is in preparation for publication elsewhere.

The Circuminsular Banks.—Many of the islands are associated with submarine banks, from 20 to 40 or 50 fathoms in depth. The largest of these extends 80 miles eastward from Porto Rico with a width of 25 or 30 miles, and bears the mountainous Virgin islands, five of which are from 3 to 11 miles in length; but not St. Croix, which rises from a much smaller bank of its own, farther south. The second largest bank is at the other end of the chain, next north of Trinidad; it has a length of 98 miles and a breadth of from 10 to 17 miles, and includes the mountainous island of Grenada as well as the little Grenadines. Other island-bearing banks will be mentioned below. Besides these, a few additional banks, mostly of moderate or small size, are charted without islands; the largest of these is Saba bank, 33 by 20 miles.

Relation of Islands and Banks.—In general, the younger a volcanic island the smaller the bank around it; the older the island, as indicated by the stage of its dissection, the larger the associated bank. Thus Saba island, near the bank of the same name, is a young, moderately clift volcanic cone, 3 miles across and 2820 feet high, with a bank less than a mile

wide around it. The long bank from which Statia, St. Kitts and Nevis rise appears to have been formed in association with the older, much degraded elements of those islands; their lofty younger cones are believed to have been built up on the bank after it had assumed nearly its present dimensions. The Virgin islands are of so ancient eruption that they are all elaborately dissected; some of them are mere skeletons; their bank is of large size. The little Grenadines are also elaborately dissected and rise from the northern part of a large bank; Grenada, on the southern part of the same bank, is of later origin and was probably piled up after the bank had been formed around small islands, like the Grenadines, of earlier origin, which are now buried beneath the new island.

Subsidence of Islands and Building of Banks.—All the maturely dissected volcanic islands have been more or less submerged since their eruptive growth was completed and while their dissection was advancing. This is shown by the embayments which enter the valleys or by the delta plains which replace the embayments. The submergence is ascribed to island subsidence and not to Postglacial ocean rise, because the embayed valleys are so maturely widened and, in a number of instances, their estimated rock-bottom depth at the bay mouth is so great that it is believed they could not have been excavated only during the Glacial epochs of lowered ocean level. Hence the banks around these islands have been built up by aggradational processes upon subsiding foundations. This makes it probable that, in the case of banks that bear no islands, the island foundations have been completely submerged.

The Islands Were Reef-Protected while the Banks Were Aggraded.—Coral reefs are at present imperfectly developed on the submarine banks, chiefly as discontinuous fringing reefs well back from the bank borders; but inasmuch as the inter-bay headlands have been only moderately and recently clift, it is believed that, during most of the period of island dissection and subsidence and hence during most of the period of bank aggradation, the islands have been protected from cliff-cutting by vigorous barrier reefs. It is only lately, and probably during the Glacial epochs when the temperature and level of the ocean were somewhat lowered, that such reef protection failed and that the headlands were clift. The fact that many of the headland cliffs plunge to a small depth below present sea level favors this view.

The Banks Represent Reef-Enclosed Lagoon Floors, Formed According to Darwin's Theory and Modified by the Processes of the Glacial-Control Theory. The Lesser Antillean banks are therefore interpreted as having been formed by the aggradation of lagoon floors that were enclosed by up-growing barrier reefs over slowly subsiding volcanic foundations, essentially according to Darwin's theory of coral reefs; but the barrier reefs are thought to have been cut off by low-level abrasion, the lagoon floors are suspected

of having been somewhat planed down, and the headlands are believed to have been moderately clift, when the corals of the protecting reefs were weakened or killed by the slightly reduced temperature of the lowered ocean in the Glacial epochs, as postulated in Daly's Glacial-control theory. The reef foundations, however, are not supposed to have been stable, as is also postulated in that theory; and the 30-fathom measure there accepted for the lowering of the Glacial ocean has not been confirmed by observation.

The Lesser Antilles Lie in the Marginal Belt of the Atlantic Coral Seas.—The absence of vigorous, bank-border barrier reefs around the island-bearing banks of today and of vigorous atoll reefs around the island-free banks is ascribed to the position of the Lesser Antilles in the marginal belt of the Atlantic coral seas, where the temperature of Postglacial time is thought to be not quite so high as that of Interglacial and Preglacial time.[1] Some of the banks have been shown by Vaughan[2] to be slightly benched at several levels, as if by abrasion in the rising Postglacial ocean.

Contrast between the Lesser Antillean Bank Reefs and the Typical Reefs of the Pacific Coral Seas.—The existing bank reefs of the Lesser Antilles, as here interpreted, are novices of recent establishment, and therefore not the direct successors of the infered bank-border, lagoon-enclosing barrier reefs of Preglacial time. The two sets of reefs—the earlier ones of large depth, the later ones of very small depth—are probably separated by a surface of abrasion which truncated the earlier established reefs, and which serves as a foundation for the island-fringing novices of today. In this respect, the novice reefs of the Lesser Antilles are unlike the veteran barrier and atoll reefs of the Pacific coral seas; for there the reefs of earlier date do not appear to have been cut away in the Glacial epochs; and the present reefs, with their exterior slopes descending into deep water, are the direct successors of the earlier ones. It is only in the marginal belts of the Pacific coral seas that novice reefs of new establishment are found, comparable to the novice bank reefs of the Lesser Antilles.[3]

Second-Cycle Islands.—The Lesser Antillean islands which are composed of volcanic and calcareous rocks, or of calcareous rocks alone, have evidently suffered uplift; they and their submarine banks are therefore at present in a second cycle of development. They appear, however, to have been formed, before their uplift, in the same manner as the islands and banks which have not been uplifted. Insofar as their calcareous areas have subsequently been worn down to lowlands and submerged, or abraded to low-level platforms in the Glacial epochs, the resulting banks may be treated as of a second generation. Several of these second-cycle islands merit brief description.

Two Uplifted Atolls.—Marie Galante, an independent member of the Guadaloupe group, is a limestone island, 9 miles in diameter, with a very even upland profile at a height of 670 feet. It is little dissected, and

appears to represent a recently uplifted bank or atoll; it is bordered by a narrow bank. Sombrero, a narrow limestone island a mile long and from 25 to 40 feet high, rising from a 3-by-4 mile bank, has been well described by Julien,[4] who interpreted it as the remnant of an uplifted atoll which was originally about as large as the present bank.

A Tilted and Partly Re-leveled Atoll.—Antigua and Barbuda, with the 30-mile bank between them, are believed to represent a tilted and partly re-leveled atoll. The atoll appears to have been tilted up in the southwestern or Antigua area, perhaps to a potential altitude of 8000 feet, and then worn down to moderate or low relief; but tilted down to a moderate depth and then built up again in the northeastern or Barbuda area. The beveled volcanic and calcareous rocks, dipping 10° or 15° northeast all across the Antigua area, have been well studied a number of geologists, most recently by Earle;[5] their reports give repeated evidence of island growth during a long period of slow subsidence, first by eruptive processes to a thickness of 5000 feet or more, later by calcareous sedimentation in Oligocene time to a thickness of at least 1500 feet. Confirmation is thus found for the inferences above stated concerning the origin of the banks that have thus far escaped tilting. The calcareous strata presumably represent lagoon deposits, at first enclosed by a barrier reef,[6] but later, when the volcanic island disappeared, by an atoll reef.

Second-Cycle History of the Antigua-Barbuda Atoll.—As the Antigua-Barbuda atoll appears to have been formed in Oligocene time, the uplift and erosion of its Antigua area presumably occupied a considerable part of later Tertiary time. After the erosion was so well advanced that the weaker calcareous beds were worn down to lowlands and the more resistant volcanic rocks were reduced to submountainous forms, the resulting island was further diminished in size by subsidence, which increased the area of the bank of second generation and embayed the island shore line; the evidence for this seond-cycle subsidence, like that for the first-cycle subsidence above described, is found in the form of the embayed valleys. During most of the time that the erosion of Antigua was in progress, it must have been, again as in the case of the first-cycle islands, protected from abrasion by encircling reefs; for here, as in the other islands, the subdued headlands between the broadly opened and partly embayed valleys are but little clift; yet here again, as in the other islands, the fact that the headlands were somewhat clift after they had gained essentially their present forms points to the occurrence of a relatively recent and brief interval, probably coinciding with the Glacial epochs of lowered ocean level and temperature, when reef protection failed. At present, novice reefs are found on the Antigua-Barbuda bank of second generation, similar to those on the banks of first generation.

Classification of the Lesser Antilles.—An ideal scheme of island develop-

ment may now be outlined in terms of which the islands of the Lesser Antilles may be genetically classified. The scheme includes one complete cycle of island history and part of a second cycle. The first cycle opens with the eruptive upgrowth of a volcanic island, presumably on a slowly subsiding foundation; as the cycle advances the island may be increased in size by new eruptions, which for a time more than make good the loss of size caused by erosion and subsidence; then after eruptions cease, the submarine slopes of the slowly subsiding and diminishing island are more and more encroached upon by accumulating lagoon deposits within an upgrowing barrier reef; and when the island is wholly submerged the barrier becomes an atoll reef enclosing an island-free lagoon. A small episode of reef abrasion and of headland cliffing occurs at whatever stage of the cycle is attained when the Glacial period is encountered. The second cycle, introduced by upheaval or uptilting, may interrupt the first cycle at any stage in its progress. The second-cycle island thus exposed to developmental changes is not a young and growing volcano, such as was formed at the beginning of the first cycle, but an uplifted barrier reef with a central island or an uplifted atoll without a central island, and is composed of calcareous strata on a volcanic foundation; it then, with or without renewed volcanic activity, runs through a sequence of erosional changes associated with subsidence and renewed reef growth, again with an episode of abrasion when the Glacial period is encountered. Evidently, an island now well advanced in the second cycle was originally formed by volcanic eruption at a much earlier geological date than an island now at the beginning of its first cycle: but as the island in the second cycle is, after its time of upheaval, again subject to subsidence, it is inferred that young, first-cycle islands are also as a rule formed during the subsidence of the ocean floor.

According to this scheme, Saba is a small young island, probably of Pleistocene eruption. The Saints, next south of Guadaloupe, are a small group of maturely dissected, partly submerged, and moderately clift volcanic residuals, 5 miles over all, rising from a well defined bank, 8 or 10 miles across. Redonda, between Nevis and Montserrat, is a small, clift residual, less than a mile in diameter, rising from an imperfectly charted bank of moderate size; Redonda is therefore taken to be an abraded, almost-atoll islet. The small island-free banks close this simple series. Another series may be made of larger islands and banks. Montserrat includes a maturely dissected cone, probably of Pliocene eruption, adjoined on its clift northern side by a 2-mile bank, and overlapped on its southern side by the nearer members of a group of much younger cones, by which the island was increased perhaps ten-fold in size; the bank around their shores is of insignificant width. As already intimated, Statia consists of a deeply dissected cone on the north, in part overlapped by a young and loftier cone on the south; St.Kitts consists of several small volcanic

residuals tied together by sand reefs and adjoined on the north by a group of lofty young cones; Nevis consists of three maturely dissected residuals, over and around which a loftier young cone has been built up; the bank 47 miles in length, above which these three composite islands rise, was probably formed as a barrier-reef lagoon floor on the flanks of the subsiding older elements of the group, and the newer cones were later built up on it. The occurrence of large slabs of limestone, some of them over a hundred feet across, on the flanks of the younger cone of Statia[7] and of Brimstone hill,[8] a small parasitic cone on a larger young cone of St.Kitts, supports this view. Dominica, Martinique, St.Lucia and St.Vincent are composite islands of larger size; their cones and mud-flows of later eruption, now moderately dissected, probably cover to a greater or less extent their earlier-formed banks as well as the still earlier erupted and much denuded smaller cones, during the subsidence of which the banks were built up. The deeply eroded and partly submerged Grenadines probably had, as already suggested, similar companions on the site of the later formed and much larger island of Grenada, the addition of which near the western side of the large bank that had been aggraded during the subsidence of the earlier islands explains the narrowness of the bank on that side. The deeply eroded and elaborately embayed Virgin islands have probably lost much of their original area by subsidence, for their bank is very large. Part of the bank foundation may have been, before any volcanic cones were built upon it, a hilly lowland of deformed and eroded continental rocks extending eastward from Porto Rico, for patches of such rocks are still visible. A slight and recent uplift at the eastern end of the Virgin bank has produced Anegada island, wholly of limestone; it is an aberrant feature, suggesting the introduction of a second cycle of island-and-bank development.

The three examples of second-cycle islands already described appear to have reached the atoll stage of their first cycle before their second cycle was opened by uplift. Marie Galante was a moderate sized atoll before its recent uplift; now it is a little dissected island-upland, surrounded by a narrow bank of second generation. Sombrero was a smaller sized atoll before uplift and has suffered a greater loss since uplift, thus giving it a wider bank of second generation. The large Antigua-Barbuda atoll was tilted long ago; and in its southwestern area of greatest uplift, it has been degraded about as much since the time of tilting as the volcanic foundation had subsided before that time; its bank of second generation is well developed. A similar history may be suggested for St.Bartholomew, St. Martin and Anguilla, with their large bank; and also for Grande Terre and Desirade and their bank; but the history of the latter group seems to have been complicated by the addition of the lofty volcanic range of Guadaloupe on the west long after the formation of an earlier volcanic foundation, on which as it subsided the limestone islands were aggraded.

The scheme on which this classification is based will doubtless need various modifications before it is complete, but it is believed to be substantially correct.

[1] W. M. D. "The marginal belts of the coral seas." *Amer. J. Sci.*, **6**, 1923 (181–195).

[2] T. W. Vaughan. "Physiographic features of the Virgin and northern Leeward islands." *J. Wash. Acad. Sci.*, **6**, 1916 (53–66).

[3] W. M. D. "Drowned coral reefs south of Japan." *Proc. Nat. Acad. Sci.*, **9**, 1923 (58–62).

[4] A. A. Julien. "On the geology of the key of Sombrero." *Ann. Ly. N. H. New York*, **8**, 1867 (251–278).

[5] K. W. Earle. *Report on the geology of Antigua.* Antigua, 1923. This includes a bibliography of 22 titles.

[6] J. C. Purves. "Esquisse géologique de l'île d'Antigoa." *Bull. Mus. hist. nat. Belg.*, **3**, 1884 (273–318). This observer recognizes that Antigua reached the barrier-reef stage (307) but does not mention the possibility of its having become an atoll.

[7] G. A. F. Molengraaff. *De geologie van het eiland St. Eustatius.* Leiden, 1886.

[8] P. T. Cleve. "On the geology of the north-eastern West India islands." *Handl. Svensk. Vetensk. Akad.*, **9**, 1871.

RADIO FOG SIGNALS FOR THE PROTECTION OF NAVIGATION; RECENT PROGRESS

By George R. Putnam

United States Lighthouse Service

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The second president of the National Academy of Sciences, Joseph Henry, then chairman of the Lighthouse Board, between 1872 and 1878, made extensive investigations of sound in its application to fog signals. In the 50 years since that time little progress has been made in meeting the defects for fog warning purposes of sound signals transmitted through the air. Recently, however, the progress in another branch of physical science, electricity, in which also Joseph Henry was a pioneer, has permitted a broader solution of the fog signal problem, which could little have been anticipated in Henry's time.

The radio fog signal, which has come into actual use within the last three years, in conjunction with the radio compass, is likely to prove one of the greatest advances yet made in providing aids for the guidance of mariners. The radio compass, also perfected in the last few years, is probably the most important navigational instrument provided for use on shipboard since the invention of the magnetic compass. The navigator is most in need of help during fog and thick weather. Of all the former aids and navigational instruments, there were available to him in time of

fog only the sound signals of uncertain range and on which he could not take accurate bearings, the magnetic or gyroscopic compass, sounding devices, and the restricted use under fog conditions of visual aids. The radio compass and radio fog signal for the first time give the navigator a means of taking accurate bearings of fixed points or of other vessels, regardless of fog or weather conditions, and at much greater distances than are possible for sight bearings even in clear weather.

The first successful radio fog signals were established by the U. S. Lighthouse Service of the Department of Commerce, at three stations near New York, placed in commission May 1, 1921, and operated since that time. There are now in operation 11 such stations in the United States, and 12 abroad. Each station sends automatically during the continuance of fog, and generally at certain periods during clear weather, radio signals having a distinctive characteristic, by which the sending station may be recognized just as a lighthouse is known by its flashes. These signals are located on light-ships or at lighthouses whose positions are shown on the charts. The transmitting apparatus heretofore in use in the United States is a spark transmitter of about 1 kilowatt power. The characteristic is controlled by a special automatic motor-driven timing switch. The frequency used is 300 kilocycles per second (wave-length 1000 meters), and a band each side of this has been set aside for this purpose. The range of usefulness originally proposed was 30 miles, but it is actually considerably greater.

The radio compass on shipboard operates on the principle that the amount of electromotive force induced in a vertical loop by an arriving electromagnetic wave depends upon the angle between the plane of the loop and the wave front. The bearing is read by placing a portion of the system rotating about a vertical axis in a position in which the minimum electromotive force is induced by the incoming waves, this being the position in which the plane of the coil is at right angles to the direction from the signal. Two types of radio compass or direction finder are now being used on shipboard. That principally employed on American vessels includes a coil aerial consisting of a number of turns of insulated wire upon a rotatable frame about 4 feet square, mounted on a verticle spindle, and provided with a pointer for determining the position of the coil, and with suitable radio receiving apparatus, including electron tubes for amplification. The readings are referred either directly or indirectly to a magnetic or gyroscopic compass. The direction finder generally used abroad includes two large loop aerials with their planes at right angles, mounted in a fixed position above the upper deck of the vessel, one loop lying in the plane of the axis of the ship. Each loop is connected with a small fixed field coil. These two field coils are also mounted with their planes at right angles, and a small search coil is pivoted to rotate within them.

The radio bearing is taken with the search coil in the same manner as with the large rotatable loop in the other type of radio compass.

About 291 vessels, including most of the large trans-Atlantic passenger vessels, are now equipped or being equipped with radio compasses or direction-finders, and these instruments are in constant and successful use. The bearings obtained under proper conditions approach in accuracy sight bearings with a magnetic compass. The progress in the utilization of this new navigational instrument is shown by the fact that 5 months ago a list of vessels equipped included only 190 names.

The radio fog signals are being operated as a part of the regular equipment of light-ships and lighthouses, and with the regular station personnel. The radio compass on shipboard is used by the navigator himself, or by the ship's radio operator under the navigator's direction. Radio bearings are utilized in navigation in the same manner as sight bearings. The radio signal may be employed as a leading mark for which to steer, or successive bearings may be taken, or if more than one signal is available cross bearings may be used. While the regular radio fog signals are more convenient for such observations, radio bearings may be taken on any sending station operating on suitable wave-length. A very valuable use is also the taking of radio bearings on other ships equipped with radio, for the purpose of avoiding collision, or of seeking a vessel in distress.

Various precautions are necessary to obtain satisfactory results in radio compass navigation, and especially is it necessary that the compass be properly constructed, installed and calibrated. Repeated calibration of the radio compass on the lighthouse tender Tulip, covering a period of 3 years, have shown substantially constant corrections. It is found that radio bearings cutting a coast line at an acute angle are subject to some distortion. As most of the signals established by the United States are on light ships they are not in general affected by this land effect. Radio bearings are also under certain conditions, subject to distortion because of so-called night effect. This seems to apply, however, rather to bearings at considerable distances on stations of high-power using longer wave-lengths than the radio beacons use. As explained later, no serious error has been noted thus far in bearings taken on the radio fog signals of this country, and it appears probable that such errors will be either small or negligible for radio signals of this type and wave-length. The simple radio compass gives only the line of direction of the signal, and hence two readings may be taken differing by 180°. This ambiguity may be readily eliminated by a sense finding attachment, temporarily coupling a supplementary fixed antenna to the coil so as to destroy its symmetry, and permit of the determination of the approximate general direction of the signal by a supplementary observation when this is necessary.

Most of the actual progress in radio fog signals and the effective use of

the radio compass on shipboard has been made within the last three years. Prior to that an improved type of radio compass was developed by the Bureau of Standards, and the Lighthouse Service with the coöperation of that Bureau carried out various tests of transmitting apparatus and installations on shipboard. Other types have been designed abroad. The development of the electron tube has been an important factor in making possible the advance in radio direction finding work.

Other uses of the directive properties of radio have been developed or suggested for the purpose of aiding navigation, but are not included in this paper. The utilization of radio in aiding navigation is at an early stage, and there is strong probability of further important progress.

Tube Transmitters for Radio Fog Signal Stations.—As stated above, the radio fog signal stations have heretofore used spark transmitters. Tests have recently been completed of two tube transmitters, constructed for the Lighthouse Service, with a view to providing a signal which may be tuned more sharply to the wave-length prescribed for radio fog signals, and to thus lessening interference, and also to providing a more powerful and economical signal. The tests were planned especially to determine whether there is any difference in direction within the ordinary limits of navigational use, between bearings taken on signals from spark and from tube transmitters, and particularly whether within those limits there is any directional distortion of the modulated continuous wave from the tube transmitter. The tube transmitters tested use two power tubes of size designated as 250 watts, with necessary accessory equipment, all mounted on a panel excepting the generator. A temporary sending station was established on a light-ship in lower New York Bay, and on this were placed both a spark and a tube fog signal transmitter, with arrangement for sending signals alternately from these. The tests were made from the lighthouse tener Tulip, which on both occasions steamed on a straight course easterly from New York for a distance of about 115 nautical miles. Frequent radio bearings were taken by several observers, using the radio compass on the tender Tulip. Tests were made on February 13 and on March 13, 1924, with different tube transmitters on the two occasions. The results are shown in tables A and B, the readings in each group being combined, and the spark transmitter bearings compared with the mean of the tube transmitter bearings before and after each group, thus eliminating uncertainties due to course of vessel and other causes. The average of the differences is for February 13, $-0.6°$ and for March 13, $+0.6°$, results within the uncertainty of observation, and indicating that there is practically no difference in the direction of radio bearings taken on signals sent with tube and with spark transmitters, within the limitations of this test. The more erratic results on February 13 were due to unfavorable conditions, a high wind of 45 miles an hour, considerable interference,

and observations taken by 5 observers, requiring changing of adjustments.

The conditions were more favorable on March 13. On this date 93 radio bearings were taken, at distances of from 25 to 115 nautical miles from the transmitting station. Comparing these bearings with the bearings taken from the course of the Tulip, as plotted from visual observation and dead reckoning, there are only 7 readings with an apparent error over 2°, and only 1 difference over 3°; the latter was 5°, the bearing being taken at a distance of 106 nautical miles at 7 P.M. with considerable interference. The signals from both tube and spark transmitters on February 13 were received on Nantucket Shoal light-ship, 202 nautical miles from the sending station, but in a somewhat irregular manner.

While the recent tests were not especially planned to develop the facts as to interference, evidence on that subject was obtained. Considerable interference had been reported from the spark fog signal transmitter located at Cape Henry, which has been sending signals during fog since June, 1923. A tube fog signal transmitter was operated at this station for about 4 hours daily, from March 19 to 28, 1924, as mentioned below. A shore radio compass station located 5 miles away, and in a position at which Cape Henry would cause maximum disturbance, reported that there was no interference whatever during this test, and a military radio station close to the light station reported that they did not know the radio fog signal was working. Improved conditions as to interference from the tube set as compared with the spark set were also reported at New York for the test of March 13.

The comparative tests of the tube and spark transmitters indicated that the former will give signals equally effective with at least one third less power than the spark transmitter.

A tube radio fog signal transmitter was placed in regular service on Ambrose Channel light-ship, at the entrance to New York, on April 23, 1924, and it is proposed to extend the use of these transmitters in the system of radio fog signal stations for this country.

Night Effect.—Although as stated above, no serious error on account of so-called night effect has been noted or reported in the use of the radio fog signals as established by this country, some preliminary systematic tests have recently been made by the Lighthouse Service to obtain more definite information on this subject. Radio bearings were taken for periods of from 1 to 3 hours covering the time of sunrise and of sunset on 9 days, from March 19 to March 28, 1924, the transmitting station being at Cape Henry Lighthouse, Virginia, and the receiving station 30 nautical miles distant at Wolf Trap Lighthouse in Chesapeake Bay. One of the same tube transmitters described in the preceding section was used, sending modulated continuous waves, with 1000 meter wave-length. On one

evening signals were sent also for a short time with the station spark transmitter at Cape Henry. A radio compass of the single coil type was mounted at Wolf Trap, and an observer there took radio bearings at frequent intervals, 10 minutes or less, during the hours specified. A summary of the results of this test is given in table C. On 4 occasions there were periods where the minimum was quite broad, reaching a maximum at one time of 40°. At these times the bearings derived from the mean of the extreme readings became slightly more indefinite. Moderate discrepancies in individual readings occurred during these periods, but the bearings did not show a tendency toward a systematic error in either direction, and the amount of the maximum deviations of the resulting bearings from the average due to this cause was only 2°, not much more than the uncertainty to be expected in bearings taken on board ship. These disturbances were very probably due to night effect, but the conclusion so far as it is justified by this test, in conjunction with the general experience reported, is that such disturbances do not seriously affect the value of radio bearings for navigational purposes, with the character of signals used in this country and with the present limitations of their use. The principal effect of such disturbances is to make it more difficult to take radio bearings, but this very difficulty serves as a warning of the need of care and repeated bearings at such times. Similar results were obtained from a series of radio bearings taken from the lighthouse tender Orchid on 21 days in March and April, 1924, the readings being taken for some time before and after sunset or sunrise. The subject is of interest, and further data is being collected on it, both here and abroad.

During the 9 day test at Wolf Trap about 260 radio bearings were taken. The discrepancy from the mean reached 3° on only two occasions, on March 19, when the receiving set was not operating properly, and on March 26 when the static disturbance was very bad, rendering it difficult to obtain bearings. The discrepancy reached 2° on only three other occasions, on March 20 when the static was also very strong, and on March 23 and March 24 when the minimum was very broad.

TABLE A

Comparative tests of radio bearings observed on signals from tube and spark radio fog signal transmitters, February 13, 1924; transmitters on Light Vessel 108, bearings with radio compass on tender Tulip. Comparison of spark transmitter bearings with mean of tube transmitter bearings before and after each group. Observations by J. T. Yates, J. L. Preston and others.

TIME FEB. 13, 1924	TUBE TRANSMITTER RADIO BEARINGS BEFORE °	AFTER °	MEAN °	AVERAGE NUMBER OF BEARINGS	SPARK TRANSMITTER RADIO BEARINGS °	NUMBER OF BEARINGS	DIFFERENCE TUBE SET BEARING—SPARK SET BEARING °
9:30–10:00 A.M.	258.7	258.7	258.7	6	257.2	6	+1.5
9:50–10:20	258.7	257.7	258.2	6	258.6	7	—0.4

10:10–10:40	257.7	258.5	258.1	3	258.3	3	—0.2
10:30–11:00	258.5	258.0	258.2	1	258.5	2	—0.3
10:50–11:20	258.0	257.5	257.8	1	258.1	4	—0.3
11:10–11:40	257.5	257.3	257.4	2	258.7	3	—1.3
11:30–12:00	257.3	257.0	257.2	2	257.5	2	—0.3
11:50–12:20 P.M.	257.0	256.5	256.8	2	256.0	1	+0.8
12:10– 1:00	256.5	255.0	255.8	2	258.0	1	—2.2
1:10– 1:40	257.5	258.0	257.8	2	260.0	1	—2.2
1:30– 2:00	258.0	258.2	258.1	2	259.0	1	—0.9
1:50– 2:20	259.2	259.7	259.4	3	258.2	2	+1.2
2:10– 2:40	259.7	262.0	260.8	2	263.0	1	—2.2
3:30– 4:00	260.0	256.5	258.2	2	262.2	2	—4.0
3:50– 4:20	256.5	257.0	256.8	2	257.0	1	—0.2
4:10– 4:40	257.0	261.2	259.1	2	257.0	1	+2.1
						Mean	—0.6

TABLE B

Comparative tests of radio bearings observed on signals from tube and spark radio fog signal transmitters, March 13, 1924; transmitters on Light Vessel No. 108, bearings with radio compass on tender Tulip. Comparison of spark transmitter bearings, with mean of tube transmitter bearings before and after each group. Observations by J. T. Yates and J. L. Preston.

TIME MAR. 13, 1924	TUBE TRANSMITTER RADIO BEARINGS BEFORE °	AFTER °	MEAN °	AVERAGE NUMBER OF BEARINGS	SPARK TRANSMITTER RADIO BEARINGS °	NUMBER OF BEARINGS	DIFFERENCE TUBE SET BEARING–SPARK SET BEARING °
10:19–11:29 A.M.	258.4	261.1	259.8	7	260.1	5	—0.3
11:12–12:27 P.M.	261.1	262.0	261.6	8	261.2	3	+0.4
1:15– 2:26	261.1	260.7	260.9	6	260.2	2	+0.7
2:13– 3:28	260.7	261.4	261.0	6	260.9	4	+0.1
3:14– 4:25	261.4	261.6	261.5	8	261.7	3	—0.2
4:14– 5:27	261.6	261.3	261.4	10	260.9	5	+0.5
5:14– 6:29	261.3	260.8	261.0	7	259.5	1	+1.5
6:15– 7:20	260.8	261.2	261.0	4	259.0	1	+2.0
						Average	+0.6

TABLE C

Tests for night effect; radio bearings, taken near sunrise and sunset, March 19 to March 28, 1924, with a radio compass at Wolf Trap Light Station, on radio fog signals from a tube transmitter at Cape Henry Light Station, Virginia (modulated continuous wave). Observations by A. W. Tupper.

DATE 1924	TIME OF OBSERVATIONS	TIME OF SUNSET OR SUNRISE	AVERAGE BEARING °	GREATEST DIFFERENCE FROM MEAN °	REMARKS
Mar. 19	5:30–7:40 P.M.	6:19 P.M.	161.8	3.0	Variations due to imperfect adjustment of receiving set.
Mar. 20	4:58–6:50 A.M.	6:15 A.M.	162.0	2.0	
Mar. 20	4:55–7:55 P.M.	6:20 P.M.	162.5	1.0	
Mar. 21	4:58–6:55 A.M.	6:14 A.M.	162.5	1.0	
Mar. 21	4:50–7:50 P.M.	6:21 P.M.	162.5	0.5	
Mar. 22	5:05–6:40 A.M.	6:12 A.M.	162.5	1.5	Very broad minimum for about 1 hour.

TABLE C (Continued)

DATE 1924	TIME OF OBSERVATIONS	TIME OF SUNSET OR SUNRISE	AVERAGE BEARING	GREATEST DIFFERENCE FROM MEAN	REMARKS
			°	°	
Mar. 22	4:55–7:45 P.M.	6:22 P.M.	162.5	0.0	
Mar. 23	5:05–6:45 A.M.	6:11 A.M.	162.5	0.0	
Mar. 23	4:54–7:52 P.M.	6:23 P.M.	162.9	2.0	Broad minimum.
Mar. 24	5:10–6:45 A.M.	6:10 A.M.	162.7	1.5	
Mar. 24	4:55–7:50 P.M.	6:23 P.M.	162.5	2.0	Very broad minimum for about 1 hour.
Mar. 25	5:05–6:50 A.M.	6:08 A.M.	162.6	1.5	
Mar. 25	4:55–7:55 P.M.	6:24 P.M.	162.5	0.0	
Mar. 25	8:00–8:35 P.M.		162.5	0.5	These readings on signals from spark transmitter.
Mar. 26	5:15–6:50 A.M.	6:06 A.M.	162.3	1.5	
Mar. 26	5:00–7:45 P.M.	6:24 P.M.	161.6	3.0	Static very bad; difficult to take bearing during entire period.
Mar. 27	5:15–6:55 A.M.	6:04 A.M.	161.7	1.0	
Mar. 27	4:55–7:58 P.M.	6:26 P.M.	162.0	1.0	
Mar. 28	5:15–7:00 A.M.	6:03 A.M.	162.4	0.5	
		For 9 days	162.3		

SELECTIVE FERTILIZATION AMONG THE GAMETES FROM THE SAME INDIVIDUALS

BY D. F. JONES

CONNECTICUT AGRICULTURE EXPERIMENT STATION, NEW HAVEN

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When certain varieties of pop corn of the *Zea mays everta* type are crossed with sweet corn a deficiency in the number of recessive segregates is commonly observed when the hybrid plants are self-fertilized. This was noted by Correns, as reported in 1901, in the first Mendelian experiments with this plant. While most varieties having corneous endosperm give the expected 3 to 1 ratio when crossed with the wrinkled translucent endosperm type known as sweet or sugary, as shown by Lock (1906) and East and Hayes (1911), a variety of pop corn having very hard and sharply pointed seeds, called Rice Pop or Squirrel Tooth, gives significantly less than 25 per cent of recessive seeds when crossed with various types of sweet corn.

Kempton (1919) reports a similar case with waxy endosperm. This character is like sweet endosperm in that it is inherited as a simple Mendelian recessive. The F_1 plants produced a small but regular and significant deficiency of recessive seeds. Pollen from the F_1 plants backcrossed on

the recessive parent showed a slight deficiency of waxy seeds which was not shown in the reciprocal pollinations.

The cross of an inbred strain of pointed pop corn with a first generation hybrid of two inbred strains of sweet corn of different type gave the results shown in table I. One of the sweet strains was a small, yellow-seeded

TABLE I

	Dominant Seeds	Recessive Seeds	Ratio of Dominants to one Recessive
(Su x su)F_2 1	366	43	8.5
2	346	94	3.7
3	376	65	5.8
(su x Su)F_2 1	322	56	5.8
2	378	77	4.9
3	421	98	4.3
4	391	47	8.3
5	485	116	4.2
Total Found	3085	596	5.2
Expected	2761	920	
Deviation	324 = 18.3 x P. E.		

variety known as Golden Bantam, the other a large, white-seeded variety known as Evergreen. Since both were homozygous for the recessive endosperm factor the somewhat complicated parentage need not obscure the simple segregation taking place between the dominant and recessive factors *Su* and *su*. This material was used because it had shown a high degree of selective action in pollen mixtures where pollen from two different types of plants acting in competition fertilized more of the ovules of its own type than of the diverse type, as previously reported in 1920 and 1922.

In a total of 3681 seeds from eight self-pollinated F_1 plants of this cross, as shown in table I, there were only 16 per cent of recessives, a deviation from the normal 25 per cent of 18 times the probable error. The ratios shown by the individual plants differed widely but were all more than 3 to 1, ranging from 3.7 to 8.5 : 1. F_1 plants of a similar cross, although of somewhat different ancestry, were backcrossed both ways with the

TABLE II

	Dominant Seeds	Recessive Seeds		Dominant Seeds	Recessive Seeds
(su x Su) x su 1	191	174	su x (su x Su) 1	135	169
2	188	199	2	123	147
3	192	222	3	88	105
4	182	162	4	133	122
5	108	134	5	99	112
6	172	158	6	168	160
7	166	148	7	139	124
8	175	200	Total found	885	939
Total found	1374	1397	Expected	912	912
Expected	1385.5	1385.5	Deviation	27 = 1.88 x P. E.	
Deviation	11.5 = 0.65 x P. E.				

recessive parent. The results are given in table II. The data show a normal one to one ratio both ways the pollinations were made. The deviations are less than one and two times the probable error. In this respect the results differ from Kempton's observations with waxy corn. In all this material the segregation of smooth, corneous seeds and wrinkled, glassy seeds was very clear-cut. Occasionally a recessive seed was found that was not deeply wrinkled and at first glance might be classed as a dominant. However, the opportunity for faulty classification was just as great in the F_1 plants backcrossed with the recessive as in the F_1 plants self-pollinated. In the one case a marked deviation was found; in the other not.

The next step was to backcross the F_1 plants with the dominant parent. Since the result of fertilization by a recessive or dominant carrying gamete would be obscured, because all the seeds would have the dominant endosperm condition, it was necessary to grow the plants and determine which were segregating and which were not. This was done by planting the backcrosses, made both ways, and allowing all of the plants to interpollinate naturally. Any plants that resulted from *Su* x *Su* fertilization would show no recessive, while most of the plants of the composition *Su* x *su* would be expected to show some recessive seeds. Without selective fertilization one half of the plants would be segregating so that one fourth of the pollen in the field would be carrying the recessive factor. Segregating plants would be expected to have half of their ovules with the recessive factor so that segregating ears should have on the average one-eighth of their seeds recessive. Selective fertilization working against the recessive gametes would reduce this proportion. Only one ear was taken from each plant and most of these contained several hundred kernels. All ears which had any sweet seeds were classed as segregating. The number of sweet seeds varied on different ears as follows; the data being taken from the backcross of F_1 pollen on the dominant parent.

NUMBER OF PLANTS	NUMBER OF RECESSIVE SEEDS
6	1 to 10
15	11 to 20
18	21 to 30
15	31 to 40
17	41 to 50
7	51 to 60
8	61 to 70
1	71 to 80
1	81 to 90
88	

It will be seen from this frequency distribution that the segregating plants which had no recessive seeds and for that reason would be wrongly classified were probably less than seven per cent and were not likely to be

enough to disturb the results seriously in view of the wide departure from the normal one to one ratio obtained. The backcross of pollen from the homozygous dominant parent on the heterozygous F_1 plants gave 207 segregating and 213 non-segregating individuals. These numbers differ from a one to one ratio less than the probable error. But the pollen from the heterozygous F_1 plants backcrossed on the homozygous dominant parent gave 88 segregating and 353 non-segregating individuals, a deviation from a one to one ratio of 18.7 times the probable error.

To sum up, the F_1 plants self-fertilized showed a selective action such that more of the gametes carrying the dominant factor united than would be the case in random mating. The heterozygous F_1 plants backcrossed with the recessive parent showed no selective action either way the pollinations were made. Backcrossed with the dominant parent there was no selective action when the pollen was alike but the pollen from the heterozygous F_1 plants showed a markedly greater fertilizing ability on the part of the gametes carrying the dominant factor.

This series of facts shows that the selective fertilization is not due to differences in the functioning of the pollen alone, although the differential action is exhibited only when the pollen is diverse, otherwise unequal numbers would be obtained in the backcross on the recessive parent as well as on the dominant. There is apparently an interaction between the pollen tube and the tissues in which it grows such that *pollen carrying the dominant factor is better able to accomplish fertilization than the pollen carrying its recessive allelomorph only in a sporophyte which also has the dominant factor either in the haploid or diploid state.*

There is in this case a tendency among diverse gametes from the same individuals for like to mate with like just as among individuals of different germinal constitution, as has been shown by the results of pollination with mixtures of pollen from different plants. For this reason a variation occurring in a homogeneous population tends to be separated from the parent stock more quickly than would occur in random mating. Gametes carrying the aberrant gene are handicapped when mating with gametes from individuals of the original type but not with those from the new form. In this case gametic selection favors divergent evolution.

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THE JAPANESE RABBIT AND GAMETIC PURITY

By W. E. Castle

Bussey Institution, Harvard University

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In the hair of most wild rodents both black and yellow pigments occur distributed in such a way that in one region of the individual hair black predominates, and in another region yellow predominates. The familiar gray coat of wild rabbits, rats and mice, is produced in this way. Usually the very tip of the hair is black, then follows a subapical zone of yellow, below which the hair again becomes black, decreasing in intensity to the base of the hair which is a dull leaden or bluish black. In many species of rodents the pigmentation of the ventral surface is less intense and the black tip of the hairs may there be wanting, so that the fur appears yellowish or whitish in surface view, as in wild rabbits, the common "cotton tail" for example.

The gray type of coat has undoubted concealment value to its possessor, as any one can testify who has looked straight at a rabbit crouching in the dead grass but without seeing it. The genetic basis of the gray coat is remarkably simple. Immediately it depends upon a single gene, the agouti factor, so-called because of its notable expression in the South American Agouti. Ultimately several other genes are concerned in producing the gray or agouti coat of rodents, but mutation in the agouti gene alone will cause disappearance of the banded pigmentation of the hairs, on which change the coat becomes black throughout. Black varieties produced by a recessive (loss?) mutation in the agouti factor alone, are known in domestic rabbits, in guinea-pigs, in rats (both *Mus norvegicus* and *M. rattus*), and in the house mouse.

Another recessive gene mutation in rodents may greatly reduce the amount of black pigment in the coat or suppress it altogether, without diminishing the production of yellow pigment. The effect of this change is in the rabbit to make the coat yellow with a white ventral surface. The white ventral surface is due to the agouti factor, and it disappears if the non-agouti mutation is associated with the yellow mutation. The coat is then yellow throughout. A yellow rabbit of this sort differs from the black variety by a single gene (*E*, *e*, the extension factor). In crosses black is dominant. But a yellow of this sort differs from gray by two factors, viz. as regards *A*, *a*, the agouti factor, and *E*, *e*, the extension factor. Such a yellow is *a e* in constitution, whereas gray is *A E*. When the two are crossed, F_1 is gray, but F_2 includes (1) gray, (2) black, (3) yellow with white belly, and (4) yellow all over.

The black variety, as we have seen, is *a E* in formula, the all yellow

variety is *a e*. The two differ by a single gene, black containing *E*, yellow *e*. A third allelomorph of this gene is found in the so-called "Japanese" rabbit, which is yellow mottled or brindled with black. Such brindled individuals are not uncommon among other tame rodents and among tame mammals generally, for example guinea-pigs, cats, dogs, cattle and swine. In the guinea-pig (Ibsen), as in the rabbit, brindle is a third allelomorph of simple black and simple yellow, being recessive to the former and dominant to the latter. But the case of the Japanese rabbit is peculiar and unique, as regards the relation of the brindling to the agouti factor. The yellow portions of the coat show the usual relation to the agouti factor, the coat being white ventrally and clear yellow dorsally, whereas non-agouti yellow rabbits are yellow all over, without white belly, but with more or less sootiness of the yellow, particularly at the extremities. But the black areas in the coat of a Japanese rabbit show no influence of the agouti factor, when present. The black hairs are black from tip to base without the agouti marking seen in gray rabbits. Likewise in the black areas of the belly the hairs are black to the tip, not white-tipped as in gray rabbits.

These facts would be quite without explanation were we not acquainted with a second black variety of rabbit in which, as in the Japanese variety, the agouti factor is practically without effect on the appearance of the coat. "Dominant black" forms in reality a *fourth* allelomorph of the extension factor which we may designate E^D. Now Japanese behaves as if it were a mosaic of the allelomorphs E^D and *e*. Wherever E^D shows its influence in the coat the hair is black, regardless of the presence of the agouti factor. Wherever *e* shows its influence in the presence of the agouti factor, the fur is clear yellow dorsally and white ventrally. How can such a peculiar type of mosaic gamete have arisen? The simplest explanation is that it arose through imperfect segregation in a heterozygote between E^D and *e*. But such occurrences are very uncommon and cannot be produced at will. Black rabbits have been crossed with yellow ones a great many times but in the cross-breds, so far as we know, segregation has always been complete, each gamete transmitting *either* black or yellow, but never both, except in one solitary case, which occurred probably about thirty or forty years ago in France, and has not been duplicated since or elsewhere. For all Japanese rabbits, so far as can be learned, are descended from a stock, specimens of which were exhibited in *Le jardin d'acclimatation*, Paris, about 1895.

The type of gamete which is responsible for Japanese retains its mosaic character with great tenacity, although the somatic expression of the character may vary greatly. Some individuals show a predominance of black in the coat, others show a predominance of yellow, but I have not observed that there is any correspondence between parents and offspring

in this regard. Two Japanese young which had originally been recorded as yellows (in a litter where yellow young were not expected to appear, but only blacks and Japanese) were found later to have a few black hairs on the under surface of the tail, being elsewhere yellow, yet they bred like typical Japanese. Another young one from this same mating, recorded as yellow, showed no black hairs whatever even to the most careful inspection, yet proved to be genetically Japanese, as it produced the usual percentage of Japanese young when crossed with yellows.

The significance of the Japanese rabbit is that it shows brindling to depend upon the formation of a mosaic type of gene, very rare in occurrence but very stable when once produced, and due probably to imperfect segregation in a heterozygote between black and yellow, which are ordinarily simple allelomorphs. The convincing evidence for this view is based on the fact that Japanese behaves like a mosaic of the rare "*dominant black*" of rabbits and the common recessive yellow.

For purposes of description, however, it will be simpler to ignore the probable mosaic origin of Japanese, and regard it simply as a fourth allelomorph of the extension factor, designating it by e^J.

THE NATURE OF SIZE INHERITANCE

By Karl Sax

Maine Agricultural Experiment Station

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The genetic analysis of size inheritance has been difficult because many factors are usually involved which cannot be distinguished in their effect. Size differences have generally been attributed to multiple factors but some investigators have questioned the Mendelian interpretation of size inheritance. If certain qualitative characters are found to be dependent on single factor differences, which behave in a simple Mendelian manner, then any factors for size in the same linkage group must also behave in a like manner.

Recent experiments on size inheritance of beans have shown that size factors are frequently associated with simple factors for seed coat color and pattern.[1] A genetic analysis of the second generation of the cross Improved Yellow Eye × Small White showed simple 3 to 1 ratios for pigmented versus white, mottled versus self-colored, and completely pigmented versus partially pigmented or eyed. Factor differences for seed weight were found to be linked with the eye factor, the pigmentation factor, the mottling factor, a factor for eye pattern, and with certain color factors. These size factors when combined have a cumulative effect.

This work has been carried to the F_3 and even more striking differences are found when larger numbers of individuals are studied. For a preliminary analysis of the data a single bean was selected from each F_3 segregate. The pattern or color, the number of beans described, and the average weight in centigrams is shown in table I. Individual bean weights were not obtained so the probable errors could not be calculated. Judging from the variability in F_2 however, the probable errors should be about .5 for 50 individuals, about .4 for 100 individuals, and about .1 for 800 individuals. From the data presented it is apparent that size differences are associated with the pigmentation factor, the extension factor, the mottling factor, eye pattern factors and color factors. Since certain of these pattern factors have been shown to be dependent on single factor differences it is evident that some of the differences in average seed weight are also dependent on size factors which are inherited in a simple Mendelian manner. The presence of several such size factors results in so-called blending inheritance.

TABLE I

RELATION BETWEEN PATTERN AND COLOR OF SEED COAT AND AVERAGE SEED WEIGHT IN F_3 SEGREGATES OF THE CROSS OF IMPROVED YELLOW EYE AND A SMALL WHITE BEAN. ONE BEAN WAS SELECTED FROM EACH F_3 PLANT

PATTERN OR COLOR	NUMBER OF BEANS	AV. WEIGHT CGMS.
White	1612	33.7
Self-colored	699	36.0
Mottled	935	37.2
Eyed	522	38.1
Eyed (pattern 1)	89	37.0
Eyed (pattern 5)	30	41.7
Mottled (yellow)	51	40.9
Mottled (purple)	81	35.9

A more complete analysis was made on certain of the F_3 segregates, to determine the relation between pattern and pigmentation with total yield, average seed weight, and number of seeds per plant. Some of the data are presented in Table II. The average yield of the mottled segregates was found to be $10.05 \pm .15$ grams per plant as compared with $8.90 \pm .16$ grams for the white segregates and $8.03 \pm .19$ grams for the self-colored. Since the mottled pattern is dependent on a single factor these differences in total yield are also dependent on one or more factors in a single chromosome. Comparing only the self-colored and mottled segregates it is evident that the factors in a single linkage group cause an increased yield of approximately 25 per cent. The increased yield of the mottled segregates would probably be even greater if all of these individuals were homozygous for the mottling factor. If factors for total yield are not associated with the pigmentation factor, then the white

segregates should be approximately intermediate in yield between the self-colored and mottled segregates because three-eighths of the white segregates will be homozygous for mottling and two-eighths will be heterozygous for mottling. Since the pigmentation factor is absent in the whites the mottling cannot be brought into expression. As indicated in Table II the average yield of the white segregates is approximately intermediate between the self-colored and mottled classes as would be expected.

TABLE II

SHOWING THE RELATION BETWEEN SEED COAT PATTERN AND YIELD, AVERAGE SEED WEIGHT, AND NUMBER OF SEEDS PER PLANT IN F_3 SEGREGATES OF THE CROSS OF IMPROVED YELLOW EYE AND A SMALL WHITE BEAN

PATTERN OF SEED COAT	NUMBER OF PLANTS	MEAN YIELD IN GRAMS	MEAN SEED WEIGHT IN CENTIGRAMS	AV. NO. SEEDS PER PLANT
White	478	8.90 ± .16	30.16 ± .24	29.35
Self-colored	241	8.03 ± .19	34.11 ± .28	24.35
Mottled	655	10.05 ± .15	35.03 ± .15	29.01

Increased yields can be obtained by increasing the number of beans per plant or by increasing the average weight of the seed. The increased yield of the mottled segregates as compared with the white segregates is due to the increased weight of the individual seeds as indicated in Table II. The number of beans per plant in this case is approximately equal but the mottled segregates have a mean seed weight considerably larger than the white segregates. On the other hand the increased yield of the mottled segregates as compared with the self-colored is due to the increased number of beans per plant.

These results show clearly that differences in size and total yield are, to some extent at least, dependent on genetic factors which are linked with factors for simple qualitative differences and are therefore inherited in the usual Mendelian manner. The association of differences in total yield with certain qualitative characters may be of value in selecting superior segregates in hybrids of economic species. In the case of the bean hybrids described the mottling factor and the factors for large yield were contributed by the small white parent, while the self-color factor and factors for relatively low yield were contributed by the yellow eyed parent. Cases have been found however, where a factor for large size was contributed by the smaller parent. This condition would be expected occasionally because the smaller variety might carry several factors for large size and yet be small due to the larger number of factors for small size. Thus certain crosses would result in transgressive segregation in F_2 and subsequent generations, so that some segregates would be expected to be more productive than either parent.

In several recent papers Castle[2,3] concludes that the genetic agencies

effecting size in rabbits are general in their action influencing in the same general direction all parts of the body. If, as Castle maintains, the same factors which control weight also completely control length of skull, length of humerus, etc., then the correlation between skull and humerus length would be zero when weight is held constant by means of a partial correlation coefficient. Any correlation greater than zero would indicate either the effect of environmental factors or hereditary factors other than those which control weight.

In pure races of rabbits or in F_1 hybrids there should be no *genetic* correlation between the lengths of the various bones. In the F_1 of Polish $\times$ Flemish, as reported by Castle, the correlation between skull and humerus length is .49 $\pm$ .10, and .47 $\pm$ .10 when weight is held constant, indicating that this degree of correlation is due to environmental causes. In the F_2 of this cross the total correlation between skull and humerus length is .70 $\pm$ .03, and with weight constant the correlation is .51 $\pm$ .05. For the total population as used by Castle the gross and net correlations are .87 $\pm$ .01 and .55 $\pm$ .03, respectively. Since with weight constant the correlation between skull and femur length in F_2 and in the total population is not significantly greater than in F_1 it appears that the factors which control weight also control the genetic differences in length of the bones considered, thus supporting Castle's conclusions so far as his rabbits are concerned.

Wright's[4] analysis of McDowell's data as well as the work of Sumner[5] and Davenport[6] would indicate however, that body proportions may not be entirely dependent on factors which affect total weight. The partial independence of body proportions and total weight is also well illustrated in the various breeds of dogs, where there is considerable variation in the relative lengths of skull and legs independently of total weight.

[1] Sax, *Genetics*, **8**, 552–560, 1923.
[2] Castle, *Carnegie Institution Publ.*, 1922 (320).
[3] Castle, these PROCEEDINGS, Jan., 1924.
[4] Wright, *Genetics*, **3**, 367–374, 1918.
[5] Sumner, these PROCEEDINGS, Nov., 1923.
[6] Davenport, *Genetics*, **2**, 313–389, 1917.

A STATISTICAL DISCUSSION OF SETS OF PRECISE ASTRONOMICAL MEASUREMENTS, II: PROPER MOTIONS

BY EDWIN B. WILSON AND WILLEM J. LUYTEN

HARVARD SCHOOL OF PUBLIC HEALTH AND HARVARD COLLEGE OBSERVATORY

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In a previous paper[1] we compared the differences between 237 photographic parallaxes[2] from Allegheny and McCormick with the probable errors of the differences and found that to bring one half of the differences within their probable errors a factor of $1.24 \pm .06$ should be applied to the published values of the probable errors. As a by-product of the determination of parallaxes the observers publish photographic proper motions in Right Ascension with reference to the mean of the comparison stars.[3] We propose now to see what light can be thrown on the question of the amplification factor for probable errors by comparing the photographic values of each of these observers (A and M) with the meridian circle determinations (Boss).

Throwing out double stars as liable to special errors, we have tabulated the differences $(B-A)$ between Boss and Allegheny and $(B-M)$ between Boss and McCormick for all remaining stars common to the lists, about 290 in each case. As the solar motion will introduce systematic differences between photographic and meridian circle determinations of proper motion, we first correct $B-A$ and $B-M$ for it in the following way. Let

$$B-A = a + b \sin \alpha + c \cos \alpha \qquad (\alpha = \text{R. A.})$$

Take the stars in 12 two-hour groups of R. A. and from the means find 12 equations for a, b, c. If the Boss motions as printed in the Allegheny publications are used we find

$$a = +0''.0032,\ b = +0''.0018,\ c = +0''.0112.$$

As these motions contain the correction $00''.032 \cos \delta - 0''.0022 \sin \delta \sin \alpha$ the values of a, b, c, for the original Boss motions, are[4]

$$a = 0''.001 \pm 0''.0009,\ b = 0.003 \pm 0''.0013,\ c = 0''.011 \pm 0.0013.$$

The inference would appear justifiable that the systematic difference (a) between the P. G. C. motions as printed and the Allegheny photographic values is only about equal to its probable error. On treating the differences $B-M$ in the same way (using the P. G. C. values) we find

$$a = -0''.002 \pm 0''.001,\ b = -0''.002 \pm 0''.0014,\ c = +0''.009 \pm 0''.0014.$$

This method of eliminating the solar motion should lead to a value of

the R. A. of the Solar Apex α_0, calculable from $\tan \alpha_0 = -c/b$. We find

$$\alpha_0 = 285° \pm 7° \text{ (Allegheny)}, \quad \alpha_0 = 258° \pm 8° \text{ (McCormick)},$$

which are satisfactory. The maximum parallactic motion in R. A. is $\sqrt{(b^2 + c^2)}$ or $0''.011 \pm 0''.0013$ for Allegheny and $0''.009 \pm 0''.0014$ for McCormick. Finally if we take 18 km. as a reasonable value for the component of the sun's motion relative to these comparison stars and parallel to the equatorial plane, the mean parallax of the comparison stars for Allegheny would be $0''.0027$ and for McCormick $0''.0023$, and these values do not appear unreasonable.[5]

If now the differences $B-A$ be corrected for the mean parallactic motion as thus determined we have certain discrepancies β_A. The root mean squares r of the Boss and Allegheny published errors may be formed and the quotients $\beta_A/r = \alpha$ calculated. The values of α are distributed normally except for a considerable excess of large values both positive and negative, particularly positive. The probable error of the distribution is about 1.6. A shorter method of attack is to plot directly the corrected differences β_A (Fig. 1) on arithmetic probability paper and read the slope of the central part of the line to determine the probable error as

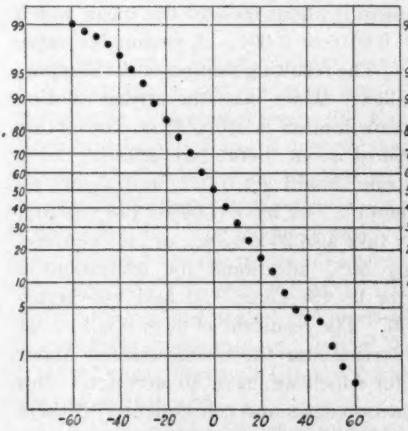

FIG. 1. BOSS—ALLEGHENY

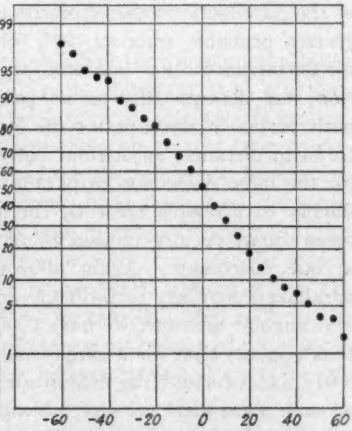

FIG. 2. BOSS—McCORMICK

Abscissae are thousandths of seconds of arc; ordinates, percentages

$0''.0145$ (the computed value from all the points is $0''.0149 \pm 0''.0006$, showing that the excess of large errors is not significant). On the other hand the mean value of the published probable error of the differences β_A is $\pm 0''.0093$ and the ratio $149/93 = 1.60$, as above. Carrying out the calculation graphically (Fig. 2) by the short method for the corrected

differences β_M between the Boss and the McCormick values, we read a probable error of 0".016 (computed value 0".0174). This is naturally larger than the figure for β_A because the McCormick published probable errors (for parallaxes, those for proper motions are not given) slightly exceed the Allegheny values, and might fairly be assumed still to represent a factor of 1.6.

It would, however, be incorrect to infer at once that for the probable errors of the photographic proper motions an amplification factor of around 1.6 (instead of 1.24 in the case of parallaxes) is needed. The corrected discrepancies β_A and β_M still contain in addition to errors of observation cosmic effects of the dispersion in parallactic motion (i.e. in distance) and of the stream motion and individual motions of the comparison stars. The question is how much do these effects probably reduce the gross factors? Let us proceed inversely and assume a factor 1.24 as determined from the more satisfactory parallax material. The cosmical effects must then account for an error of about $\sqrt{(2.56-1.54)} = 100\%$ of the errors of observation, say about 0".009 for Allegheny and 0".010 for McCormick.

Consider one comparison star. If we assume with Kapteyn[6] that the Briggs logarithms of the actual secular parallaxes follow a normal error curve with probable error 0.17 we find the root mean square of the departure of the parallactic motion from its mean is about 60% of the mean which gives a probable error of 40% of 0".010, or 0".004. A reasonable value for the stream motion[7] is 20 km./sec. The resulting motion of the comparison star depends also on its parallax. If we take the stream motion with vertex in the equator the stream motion is $20/\sqrt{2}$, or, converting to mean distance, an angular motion of about 0".008, but allowing 50% for the large dispersion in parallax one should get 0".012, and again reducing to probable error by the factor $^2/_3$ we have 0".008. For the root mean square peculiar motion we may take also 20 km./sec. or 11.5 km./sec. in one coördinate. Again allowing 50% additional for dispersion in parallaxes, reducing to probable error by the factor $^2/_3$, and converting into angular measure, we have 0".006. The resultant of 4, 8, 6 is 11. It thus appears that for a single comparison star the cosmic error is about 0".011, i.e., of about the magnitude for which we have to account. But on each plate there are 4 or 5 comparison stars and consequently, if these may be taken as a fair sample, the cosmic error should be reduced by a factor of 2 and thus we should have accounted for only $^1/_2$ of this residual discrepancy.

It would therefore seem likely that, unless the stream and peculiar motions or the dispersion in parallax are decidedly higher than values now considered fair, or unless the comparison stars are ordinarily closely related, the indications are that a factor somewhat larger than 1.24, certainly smaller than 1.6 and perhaps in the neighborhood of 1.4, should be applied

to the probable errors of photographic proper motions. We hope to return to this problem as soon as the probable errors of the McCormick proper motions are available, as we believe they soon will be.

[1] THESE PROCEEDINGS, **10**, 129 (1924).

[2] By an unfortunate transposition the number was stated as 273 instead of 237.

[3] That a comparison between these relative proper motions and the absolute motions deduced from meridian circle observations will yield results of great importance had been pointed out by Schlesinger (*Publ. Alleg. Observ.*, **4**, Introduction, 1918) and Kapteyn (*Bull. Astron. Inst. Netherlands*, No. 14, 1922). Van de Kamp (*Ibid.*, No. 29, 1923) has discussed all the available material for the purpose of determining the secular parallax of the comparison stars.

[4] The assigned probable errors were derived from the value $\pm 0''.0149$ given below.

[5] Cf. *Publ. Astron. Lab. Groningen*, No. 29, Table 26.

[6] *Ibid.*, No. 8.

[7] *Mt. Wilson Contr.*, No. 229.

THE INFLUENCE OF ALCOHOL ON DURATION OF LIFE[1]

BY RAYMOND PEARL

SCHOOL OF HYGIENE AND PUBLIC HEALTH, JOHNS HOPKINS UNIVERSITY

Read before the Academy, April 30, 1920

The problem of the effect upon duration of life of the use of ethyl alcohol as a beverage is still an open one. There is abundant evidence that *excessive* use of alcohol shortens life, but because of the difficulty of getting critically adequate data, it has not been possible up to this time to determine whether the consumption of alcoholic beverages *in moderation* has any effect upon life expectancy, or if it has what the nature and sense of that effect is. The evidence which has been adduced as bearing on the point, chiefly from the records of insurance companies, can easily be shown and repeatedly has been shown, to have no probative significance so far as the problem just stated is concerned. The case has been well put in the concluding paragraph of Chapter X of the second edition of the report[2] of the Medical Research Council's Alcohol Investigation Committee (p. 148), where it is said:

"We repeat that it does not follow from these conclusions that a consistently moderate use of alcoholic beverages shortens life, since we cannot sort out from the data brought into comparison those instances in which the habitual abuse of alcohol produces disease or death, or in which accidental abuse has a like consequence, nor can we exclude many other social factors operating for or against longevity. . . .What we have not proved and, from the evidence before us, cannot prove, is that the moderate use of alcoholic beverages, employing the word moderate as it is ordinarily used in everyday life, has any effect at all, one way or the other, upon longevity."

In 1922 I[3] prepared a preliminary report upon a collection of data which

may fairly be said, I think, to have unique value for the study of this problem. In brief this material consists of records of certain pertinent data in the life histories of many thousands of persons belonging to the working class population of Baltimore. These records were collected by trained eugenic field workers, who visited the families and by long-continued and patient investigation got the facts with a high degree of accuracy. Especial attention in these inquiries was devoted to the habits of each individual, throughout life, in respect of alcohol and tobacco. I have given elsewhere (*loc. cit.*, pp. 237–246) a detailed account of the material, its mode of collection, the care taken to ensure its critical accuracy, etc., and these details need not be repeated here.

It will suffice to state here that for a group of over 6000 individuals we have *accurate* information covering the following points:

1. *The Alcoholic Habits throughout Life.*—By this is meant that we have that sort of information which one would have if, starting with n individuals at birth, the life history of each one was individually followed, and it was recorded when he (or she) began to drink, and what and how much was consumed on the average; this process continuing till a certain absolute time (or date), at which time some of the individuals would still be alive and of known age, while the remainder would have died *at known ages and of known causes.*

2. *The Race Stock.*—Only white persons are included in the material. In the case of both American-born as well as foreign-born individuals careful investigation has been made of the race stock or stocks from which they originated.

3. *The Social and Economic Status.*—All of the persons included in the material belonged to what may be called fairly the working class population. Family income is recorded, and has been dealt with in the preliminary report cited above.

The preliminary report on this material has been criticised on two grounds: (*a*) that the classification adopted relative to grades of alcohol consumption was a poor one, not in accord with common usage, and therefore incapable of throwing light on the important question at issue; (*b*) that the actuarial methods used were unsound, in that only the dead were considered. It is believed that in the report now presented both of these criticisms have been completely met.

The material has been re-classified into the eight groups relative to drinking habits which are shown in Table 1. In this table is also shown the magnitude of the experience in the several classes, in the form of person-years exposed to risk of death in the whole experience.

As to the categories the following may be said. An abstainer, in this classification is a *total* abstainer who has never used alcohol in any form. A moderate drinker, is one who uses alcohol in any form (beer, wine or

spirits), but in small amount at any one time, *and never enough to become intoxicated*. The persons put in the moderate, steady category are those whose habits are precisely of the sort described in the British Committee's report (*loc. cit.*, p. 141): "A daily pint or two of beer, or a daily bottle of claret, or a few glasses of whiskey and soda." The moderate occasional class includes those who drink to this amount, but not as frequently or regularly as every day. The moderate unspecified class includes those of whom it was definitely known that the *amount* consumed fell in the moderate category, but regarding whom the record was incomplete or indefinite as to frequency. The heavy class includes those persons who are positively known *to have been in the habit of getting drunk*, either as an occasional spree, with abstinence in the intervening periods (heavy, occasional, otherwise abstainers); or with moderate drinking as above defined in the intervening periods (heavy, occasional, otherwise moderate); or regularly and frequently (heavy, steady); or without precise record of the frequency of intoxication (heavy, unspecified).

TABLE I

SHOWING THE TOTAL YEARS OF EXPOSURE TO RISK IN THE SEVERAL CLASSES OF DIFFERENT DRINKING HABITS

DRINKING HABITS	TOTAL YEARS EXPOSED TO RISK	
	MALES	FEMALES
Abstainer	20,950.0	41,149.0
Moderate in amount, unspecified as to frequency	10,357.0	1,739.0
Moderate in amount, occasional	12,210.0	11,288.5
Moderate in amount, steady	19,527.0	7,759.0
All moderate	42,094.0	20,786.5
Heavy in amount, unspecified as to frequency	11,746.0	867.5
Heavy in amount, occasional, otherwise abstainer	1,057.0	48.0
Heavy in amount, occasional, otherwise moderate	3,768.5	186.5
Heavy in amount, steady	9,539.0	652.5
All heavy	26,110.5	1,754.5
Total experience	89,154.5	63,690.0

I believe that this classification removes the criticism of the British report. Surely it is in accord with common usage to call a person who gets drunk a heavy drinker. It is also common usage to call a person who never gets drunk a moderate drinker. Personally I am somewhat of the opinion that the classification of Table 1 has a less strong purely physiological justification that did the one used in the prior report. But I yield to the views of my critics on this point; perhaps the more gracefully because, as the subsequent analysis will show, it makes no essential difference to the results which classification is used.

Table 1 shows that the experience is large enough to yield trustworthy

results. Over 150,000 person-years exposure to risk is a respectable total.

The material is, from an actuarial point of view, *precisely* comparable and essentially the same as the experience of a life insurance company. If such a company is ever justified in calculating a life table from its own experience with its own policy holders, living and dead (and I am not aware that anyone has ever denied the validity of such procedure) then it is justifiable to calculate life tables from the experience here under review. In one respect the present material is actuarially better than that of any insurance companies, because here there is no necessity to correct for "lapses of policies" (i.e., withdrawal from the exposed to risk group). Here a person goes out of the exposed to risk group only because of death.

It is impossible for lack of space to go into all the details of actuarial procedure followed in calculating the life tables from this material. This will be done in the final complete report on the investigation, which will be published in book form. The following skeleton outline must suffice here.

The central death rates are based on a five year grouping of the raw data, the groups being centered at 20, 25, etc., years to avoid any disturbing influence of the usual tendency to concentrate on multiples of five in the statement of age. The years of exposure to risk in each quinquennial period were obtained as follows:

5 (Sum of living and dead for all years after end of period)
+4.5 (Sum of persons living and dying in last year of period)
+3.5 (Sum of persons living and dying in fourth year of period)
+2.5 (Sum of persons living and dying in third year of period)
+1.5 (Sum of persons living and dying in second year of period)
+0.5 (Sum of persons living and dying in first year of period)

The total deaths in each period divided by the years of exposure to risk in the same period give m_x, the central death rates, from which the q_x's were obtained by the usual formula,

$$q_x = \frac{2m_x}{2 + m_x}$$

The q_x values were graduated by standard curve fitting procedures customarily used in this type of work. From the q_x values were computed in the usual way the other important life table functions l_x, and $\mathring{e}_x$.[4]

The definitive life tables were computed as of 100,000 persons entering the experience at age 30. The q_x's were actually computed from age 20 on. In the nature of the case, however, it would be unfair to the abstainers to start the tables at age 20. The reason is that the deaths prior to about age 30 include a number of persons who were actually abstainers up to the time they died but who, if they had lived, would have become drinkers,

and passed out of the abstainer into one or another of the drinking classes. The result is to make the abstainers' death rates spuriously high at the early ages. But a careful analysis of the present material, backed by common sense experience on the point, shows that by the time age 30 is reached substantially all persons who are ever going to leave the total abstainer group and pass into a drinking group have already done so. From age 30 on all groups are justly comparable in respect of their mortality.

TABLE II

MEAN AFTER LIFE TIME ($\overset{\circ}{e}_x$) OF MALE PERSONS ATTAINING THE AGE x AND HAVING THE SPECIFIED HABITS AS TO THE CONSUMPTION OF ALCOHOLIC BEVERAGES

AGE x	ABSTAINERS	ALL MODERATES	ALL HEAVIES	MODERATE STEADY
30	37.11	37.71	29.60	40.00
35	33.51	34.21	26.31	36.20
40	29.91	30.56	23.67	32.34
45	26.29	26.88	21.29	28.48
50	22.62	23.29	18.99	24.69
55	18.88	19.91	16.69	21.05
60	15.05	16.81	14.41	17.60
65	11.57	14.02	12.21	14.41
70	8.97	11.56	10.13	11.55
75	7.08	9.39	8.24	9.06
80	5.61	7.43	6.52	7.00
85	4.34	5.65	5.00	5.35
90	3.30	4.20	3.75	3.99
95	2.45	3.03	2.74	2.90
100	1.77	2.12	1.95	2.04

TABLE III

MEAN AFTER LIFE TIME ($\overset{\circ}{e}_x$) OF FEMALE PERSONS ATTAINING THE AGE x AND HAVING THE SPECIFIED HABITS AS TO THE CONSUMPTION OF ALCOHOLIC BEVERAGES

AGE x	ABSTAINERS	ALL MODERATES	ALL HEAVIES	MODERATE STEADY
30	37.94	40.93	23.93	42.15
35	34.37	37.03	20.09	38.22
40	30.75	33.02	17.12	34.23
45	27.09	28.92	14.79	30.21
50	23.45	24.87	12.93	26.26
55	19.96	21.01	11.40	22.48
60	16.71	17.44	10.12	18.96
65	13.79	14.27	9.02	15.77
70	11.22	11.53	8.00	12.94
75	9.00	9.19	6.99	10.44
80	7.08	7.19	5.80	8.20
85	5.40	5.49	4.47	6.22
90	4.03	4.08	3.39	4.59
95	2.92	2.96	2.51	3.29
100	2.05	2.08	1.81	2.27

In the final report complete life tables for each year of life from 30 on will be presented. Here because of lack of space, I can only present abbreviated tables, picking out the values at 5 year points from the complete tables for this purpose. Only one function will be here presented, namely $\mathring{e}_x$, the complete expectation of life at age x, or as it may better be put, the mean after life time of persons attaining the age x. Even for this simple function, only the more important groups can be discussed here.

From Tables 2 and 3 certain points emerge with great clarity. These are:

1. The material as a whole is normal from an actuarial viewpoint. Taking the entire experience together we have:

Mean after life time at age 30, males = 35.19 years
Mean after life time at age 30, females = 38.53 years

Remembering that the material is derived exclusively from the white population we may compare these values with the following from Glover's United States life tables.[5]

Mean after life time at age 30:
White males, Original Registration States, 1910 = 34.87
Mean after life time at age 30:
Native white males, Original Registration States, 1910 = 35.61
Mean after life time at age 30:
White females, Original Registration States, 1910 = 36.96
Mean after life time at age 30:
Native white females, Original Registration States, 1910 = 37.81

It is obvious from these values that the present material, taken as a whole, is in respect of its mortality an extremely good sample of the general population of the United States, as indicated by the Original Registration States in 1910. The results of this study cannot, therefore, be discarded or criticised on any ground of abnormality of the material.

2. At every age from 30 to 100 inclusive the persons in the "all moderate" class of drinkers, whether males or females, have a somewhat *higher* expectation of life than do the persons in the "abstainer" class of the same age. The two life curves are not widely different, but what advantage there is lies with the moderate drinkers.

3. Male persons in the "all heavy" class have a markedly *lower* expectation of life at all ages from 30 to 100 inclusive than do male persons in the "all moderate" class; and a markedly *lower* expectation of life than persons in the "abstainer" class up to and including age 60. From age 60 on the expectation of life is somewhat though not greatly higher in the "all heavy" class than in the "abstainer" class. This result is believed to be possibly due to a selective effect of the very high mortality in the heavy drinking group prior to age 60. In the case of the females the expectation of life is markedly *lower* for the persons in the "all heavy" group than for

persons in either the "abstainer" or the "all moderate" group, at all ages from 30 to 100 inclusive.

4. The "moderate steady" class, which is from certain points of view the most important one, shows consistently the *highest* expectation of life at all ages from 30 to 100 inclusive in the case of the females; and also in the males, except that after 70 the "all moderate" male group has very slightly higher expectations than the "moderate steady" male group.

These conclusions are drawn from what is believed to be demonstrably the most critically adequate material, considering both quality and quantity, which has ever been available for the study of the problem of the influence of alcohol upon the duration of human life.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University. No. 102.

[2] *Alcohol: Its Action on the Human Organism*. Second Edition. (Revised) 1924. Pp. xx + 170. London (H. M. Stationery Office).

[3] Pearl, R. Alcohol and Mortality. In *The Action of Alcohol on Man*, by E. H. Starling. London (Longmans, Green & Co.) 1923.

[4] For the laborious arithmetic involved in the construction of these life tables I am indebted to Dr. John Rice Miner.

[5] Glover, J. W. *United States Life Tables* 1890, 1901, 1910, and 1901–1910. Washington (Bureau of the Census) 1921.

TWO EXTRAORDINARY LARVAL MYRMECOPHILES FROM PANAMA

BY WILLIAM MORTON WHEELER

BUSSEY INSTITUTION, HARVARD UNIVERSITY

Communicated, April 10, 1924

The two larvæ described in the following pages both belong to the order Diptera and were taken in the nests of two different Dolichoderine ants on the same day, March 25, 1923, in the same locality, Frijoles, C. Z., in a piece of jungle less extensive than a city block and not more than two miles from the recently erected tropical laboratory of the National Research Council on Barro Colorado Island. The first larva (Fig. 1*a*) was found in the dead petiole of a Panama hat palm (*Carludovica palmata* Ruiz and Pavon) in the midst of a flourishing colony of *Dolichoderus* (*Hypoclinea*) *championi trinidadensis* Forel var. *tæniatus* Forel. Though recognized as a larval Microdon it was at once seen to be quite unlike any described species in color and ornamentation. It measured only 5.7 mm. and was broadly and regularly elliptical, with flattened creeping-sole and feebly convex dorsal surface, like the well-known Microdon larvæ of

temperate America and Europe, but the integument was smooth and of a pale blue color, with the band of minute papillæ bordering the creeping-sole carmine red. The dorsal surface bore regular longitudinal rows of large, snow-white, spoon-shaped scales, which under a high power were very finely, transversely regulose. They were largest on the mid-dorsal region, attached to the integument by rather short, thick stalks and curved backward gracefully, with blunt tips. The posterior end of the body bore the usual keg-shaped projection with the spiracles on its truncated terminal surface. This exquisite creature resembled nothing so much as a minute nudibranch mollusk, and vividly recalled the fact that just a century ago (1923, 1924) von Heyden[1] and Spix[2] had described the larva of the common European *Microdon mutabilis* Meigen as a mollusk.

The larva was carefully separated from the ants and in the laboratory transferred to a Petri dish. After creeping about for a few days it came to rest in the middle of the dish and on the morning of April 1, was found to have pupated during the night. It now had the appearance of Fig. 1*b*.

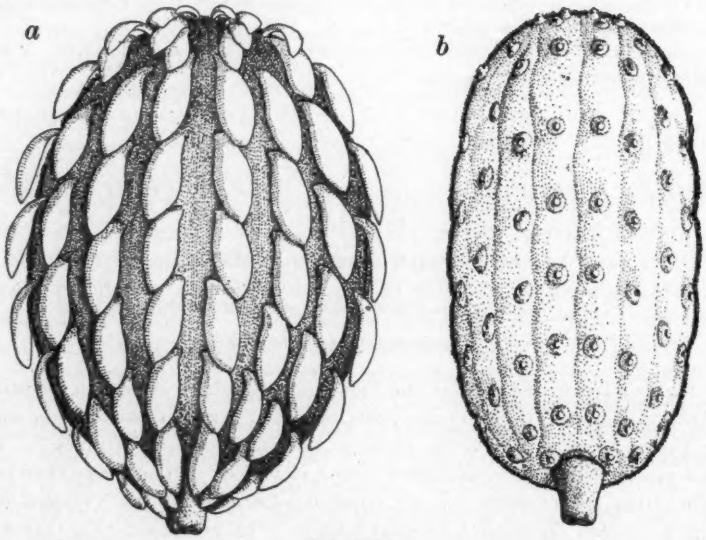

FIGURE 1

Microdon œolidiformis sp. nov. *a*. Full grown larva, dorsal aspect; *b*, *puparium* after throwing off the white scales.

The body had contracted, had become much narrower and longitudinally ridged and the integument had hardened and taken on a golden brown tint with purplish peripheral border. Apparently as a result of the strong and sudden contraction of the integument during pupation, the white

scales had been violently thrown to a distance of four to five centimeters from the insect. Rows of small boss-shaped elevations remained to indicate their former insertions. There were no traces of the prominent horn-like prothoracic stigmata seen in so many Microdon puparia, but in other respects the puparium was not unlike that of many other species of the genus. Although it was kept for some weeks under what seemed to be favorable conditions the imaginal fly unfortunately failed to emerge. I nevertheless venture to name the insect, at least provisionally, **Microdon æolidiformis** sp. nov.

The genus Microdon is represented by an unusual number of diverse and interesting species in the neotropical region. As there are very few records of their larvæ I append a brief list of those that have come under my observation:—

1. Many years ago (1901)[3] I called attention to some small flat Microdon larvæ which I found in a nest of *Pseudomyrma gracilis* Fabr. subsp. *mexicana* Roger in a hollow acacia limb near Cuernavaca, Morelos, Mexico.

2. During the summer of 1920 Mr. G. Inness Hartley and Dr. Alfred Emerson, at my request, secured one of the huge carton nests of *Azteca trigona* Emery subsp. *mediops* Forel, which are common on the trees in the second-growth jungle at Kartabo, British Guiana, and made a census of its inhabitants. The nest had a volume of 29 cubic dcm. and was estimated to contain 377,000 ants and as many larvæ and pupæ together with many myrmecophilous Clavigerid and Staphylinid beetles, Chalcididæ, Lepismidæ, etc. and a number of peculiar, large Microdon larvæ, with flat or concave dorsal and very convex ventral surfaces and therefore adapted to living on the very concave walls of the nest chambers. The imaginal fly reared from one of these larvæ by Dr. Emerson will be described by Dr. Geo. C. Wheeler.

3. During the same summer I found in a hollow twig at Kartabo a flourishing colony of a Crematogaster (near *brevispinosa* Mayr) containing four singular Microdon larvæ. They were pure white, cylindrical, caterpillar-like and covered with rather long tentacles. Unfortunately the ants and their guests had been blown out of the twig into a vial of alcohol, so that the larvæ could not be reared. They will also be described and figured by Dr. Wheeler.

4. At Red Tank, C. Z., March 10, 1923 I took from a caulinary swelling of an interesting myrmecophyte (*Cordia gerascanthus* Jacq.) a small colony of *Crematogaster brevispinosa* Mayr subsp. *tumulifera* Forel containing four small, smooth, amber-yellow, cylindrical Microdon puparia and by March 27–28, succeeded in rearing the imagines, which were scarcely larger than house-flies. They will be described by Dr. W. M. Mann.

5. During March and April 1923 Mr. James Zetek and I repeatedly

observed in various localities in the Canal Zone small narrow Microdon puparia of the usual type in the chambers of termitaria occupied by ants (*Camponotus* (*Neomyrmamblys*) *novogrenadensis* Mayr). These ants regularly take possession of the chambers adjacent to the tree trunk supporting the termitarium and permit the termites to inhabit the remainder of the structure.

6. At Las Sabanas, in the Panamanian Republic, I found small slender puparia of a Microdon very similar to the preceding in a most unusual situation, the thorns of bull-horn acacias (*A. penonomensis* Safford) occupied by the typical *Pseudomyrma gracilis* Fabr.

The nature of the food of the Microdon larvæ in the ant nests was for a long time problematical till Donisthorpe[4] succeeded in proving that the British species, *M. mutabilis*, which lives with ants of the genera Formica, Lasius or Myrmica, is a scavenger and feeds on the minute pellets expelled by its hosts from their infrabuccal pockets. More recently Borgmeier[5] has shown that certain Microdon larvæ living with the common Brazilian fire-ant, *Solenopsis sævissima* F. Smith, attack and devour the coccids which the ants keep as milch cows in their nests. I believe that the Panamanian species which I took from the caulinary swelling of *Cordia gerascanthus* may have similar habits, but the other species cited above, including *M. æolidiformis*, must feed on refuse, like the European and nearctic forms.

The other larva from Frijoles is much more remarkable than the Microdon guest of *Dolichoderus tæniatus*. I found suspended from the branch of a large tree that had just been felled by the negroes a rather small carton nest of the typical *Azteca trigona* Emery. It was about a foot broad through its base and about ten inches long. I hastily cut off the greater portion of the structure and enclosed it with its infuriated inhabitants in a cloth bag, which was taken to Mr. Zetek's laboratory in Ancon and on the following day placed in a large jar with some chloroform. I was then able to break the nest into small fragments and to examine its contents in comfort. Besides thousands of Azteca workers and their brood I found in the chambers a single small wingless Braconid myrmecophile and more than a hundred small, hard, rather shining, dark brown, lenticular objects which I at first took to be seeds. They were scattered through the chambers of all parts of the nest and were lying freely among the brood. More careful examination showed that they were Dipteron larvæ, all in the same apparently adult stage of development and of very nearly the same size (3.5–4 mm. long, 2.6–2.8 mm. wide). Unfortunately none of these larvæ recovered from the effects of the chloroform so that I was unable to rear them.

Fig. 2 shows the dorsal and ventral aspects of the insect under a low magnification. The body consists of a larger, broad, flattened carapace-

like, suboctagonal abdominal and a smaller, narrow, anterior cephalothoracic portion, which can be drawn into the abdomen. The dorsal surface of the latter is feebly convex, with two rather deep, transverse grooves near the middle and a small rounded tubercle at the posterior end. The ventral surface is flat, with a pair of longitudinal, lateral furrows, or impressions which widen posteriorly and nearly meet at a small circular area and pit, which represent the anus. The anterior portion of the abdomen above is divided by T-shaped sutures into a median triangular plate and two lateral suboblong plates, each of which bears a

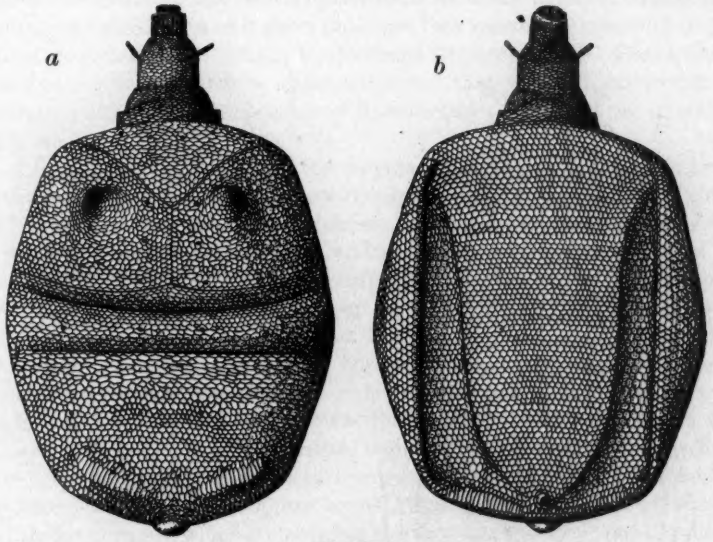

FIGURE 2

Nothomicrodon aztecarum gen. nov. et sp. nov. Full grown larva; *a*, dorsal, *b*, ventral aspect, with anterior segments extended.

clearer, thinner, slightly elevated, elliptical area which evidently represents the point of future extrusion of the prothoracic stigmal horn of the pupa. One of the lateral plates is shown under a somewhat higher magnification in Fig. 3*a*. De Meijere[6] and Metcalf[7] have found that in the Syrphidæ and closely allied families of Diptera the region of the larva bearing the two preformed, thin areas for the extrusion of the pupal prothoracic stigmal horns is the first abdominal segment. The T-shaped sutures, like the similar structures in the late Microdon larva and puparium are, of course, a prearrangement for the emergence of the adult fly.

The anterior portion of the body, shown more clearly in Fig. 3*a* as seen from the ventral side, consists of the retracted pseudocephalon with the

mouth-hooks projecting from its orifice but without distinct traces of antennæ, and three short segments representing the thorax. The first of these bears a pair of cylindrical appendages, the prothoracic spiracles. The remaining segments bear no appendages and are readily telescoped into one another when the anterior portion of the body is retracted.

The dark brown abdominal integument is very tough and leathery, neither thinner nor more flexible on the ventral than on the dorsal surface and consists of a beautiful mosaic (more regular than in the figures) of usually hexagonal chitinous plates, which tend to disintegrate when boiled in caustic potash. On both surfaces very near the posterior end of the

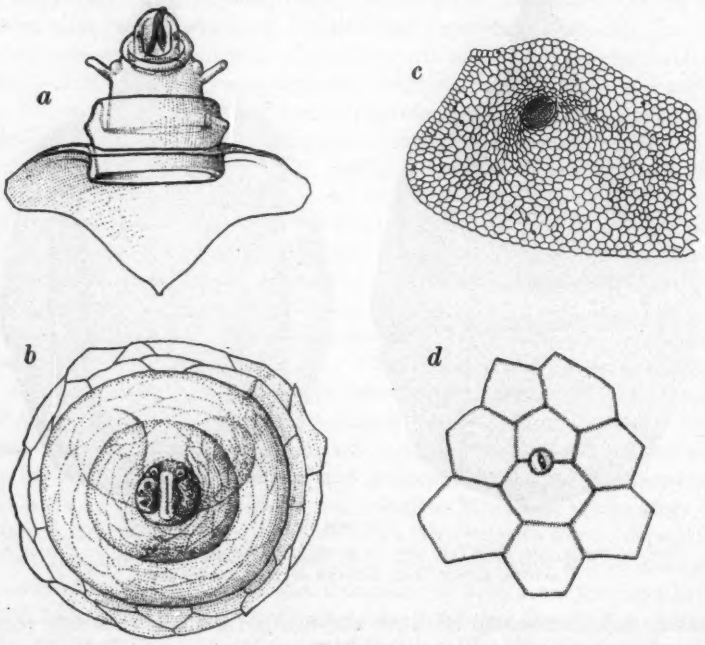

FIGURE 3

Nothomicrodon aztecarum gen. nov. et sp. nov. *a*, Pseudocephalon and thoracic segments from below; *b*, papilla at posterior end of body, showing stigmal plates; *c*, left half of first abdominal segment bearing the elliptical, finely areolated region for the extrusion of the prothoracic stigmal horn of the pupa; *d*, a sensilla from the dorsal surface.

body there are two transverse areas, each consisting of a series of narrow, elongate plates. The dorsal surface also has a number of scattered but regularly arranged sense organs (shown as black dots in Fig. 2*a*), each lying between two contiguous polygonal plates and having the appear-

ance of Fig. 3*d* under a higher magnification. These organs seem to be Hicksian sensillæ and to be the same as those seen in many other Dipteron larvæ, notably in Itonidæ (Cecidomyidæ), Syrphidæ (Microdon) and Lonchopteridæ (de Meijere[8], Maria Andries[9]). The integument of the pseudocephalon and thoracic segments is much thinner, more yellowish and consists of smaller polygonal plates than that of the abdomen. The stigmal areas on the first abdominal segment are finely areolated (Fig. 3*c*). The posterior tubercle (Fig. 3*b*) bears a pair of very small reniform stigmal plates, separated by a slit-like structure, the significance of which is by no means clear. Each stigmal plate has four small triangular spiracular openings and in front of it lies a structure that seems to correspond to the small "circular plate" of many Cyclorrhaph Dipteron larvæ. The cephalopharyngeal apparatus is so extremely small that I have been unable to obtain a satisfactory knowledge of its structure. The internal structure has not been studied.

I find it impossible to refer this singular larva to any of the recognized families of Diptera, and two accomplished specialists in this order, Prof. C. L. Metcalf and Mr. C. W. Johnson, who have examined some of my specimens, confess to the same embarrassment. There would seem to be little doubt that the creature must belong to the section Cyclorrhapha Aschiza of Brauer.[10] In this group he included four families, the Syrphidæ, Pipunculidæ, Phoridæ and Platypezidæ, and more recently de Meijere[8] has added a fifth, the Lonchopteridæ. But the larvæ of all these families are very different from the one described above. It certainly shows a vague kinship to the Syrphid Microdon, especially in the shape of the abdomen, the T-shaped sutures of its first segment, the plates for the extrusion of the prothoracic stigmal horns of the pupa, the caudal tubercle with its closely approximated stigmal plates, and the areolation of the integument, which is somewhat like that described by Maria Andries[9] for Microdon. On the other hand the rigidity of the integument on the ventral surface and absence of a creeping-sole, the proportionally much greater development of the thoracic segments, the large, cylindrical and undoubtedly functional prothoracic stigmata, the finer structure of the posterior stigmata, etc., are all characters which separate the larva under discussion from the Syrphidæ and other known aschizous Cyclorrhapha. Since it in all probability represents a new genus and may even represent a new family of Diptera, I propose to call it **Nothomicrodon aztecarum** gen. nov. et sp. nov.

What the larvæ do in the carton nests of *Azteca trigona* must remain a mystery till they are again encountered by some observer who can study the behavior of both ants and guests in an artificial nest. The powers of locomotion of the larvæ must be *nil* or limited merely to slowly dragging themselves about by means of their feeble mouth-hooks. Per-

haps they are actually carried about the nest by the ants. That they may feed on the ant larvæ is suggested by the fact that the brood was much less abundant in the nest in which they occurred than in several uninfested nests of the same ant which I examined in the avocado orchard of Mr. John English at Frijoles.

Perhaps I may be censured by some of my confrères for giving scientific names to larval insects, whose imaginal forms are presumably unknown and likely to remain so for many years, but there is good precedent for my procedure. To cite only one example, Silvestri[11] has recently named a long series of very interesting larval termitophiles. In these as in the case of the immature Diptera described above, the larval is far and away the most significant stage of the insect from an ethological or ecological point of view. Moreover, failure on my part to name the two peculiar larval myrmecophiles would sooner or later almost certainly tempt some other entomologist to tack names of his own to my descriptions—and we are advised not to lead others into temptation.

[1] von Heyden, C. H. G. "Ueber ein sonderbar gestaltetes Thierchen." *Isis*, **12** and **13**, 1823, p. 1247–1249, 1 fig.

[2] Spix. "Ueber eine neue Landschneckengattung (Scutelligera Ammerlandia)." *Abhandl. K. bayr. Akad. Wiss.*, **9**, and *Denkschr. K. Akad. Wiss. München, Math. phys. Cl.*, **19**, 1823 and 1824, p. 121–124.

[3] Wheeler, W. M. "Microdon Larvæ in Pseudomyrma Nests." *Psyche*, July 1901, p. 222–224, 1 fig.

[4] Donisthorpe, H. S. J. K. "Myrmecophilous Notes for 1911." *Ent. Record*, **24**, 1912, p. 4–10, 34–40.

[5] Borgmeier, T. "Beitrag zur Biologie der Feuerameise und ihrer Gäste (Solenopsis geminata sævissima Sm.)." *Zs. Deut. Ver. Wiss. Kunst Sao Paulo*, **3**, 1923, p. 1–9, 1 fig.

[6] de Meijere, J. C. H. "Ueber die Prothorakalstigmen der Dipterenpuppen." *Zool. Jahrb. Abth. Anat.*, **15**, 1902, p. 625–692, 4 pls.

[7] Metcalf, C. L. "The Syrphidæ of Ohio." *Ohio Biol. Survey*, **1**, 1913, 122 p., 11 pls.

[8] de Meijere, J. C. H. "Ueber die Larve von Lonchoptera." *Zool. Jahrb. Abt. Anat.*, **14**, 1900, p. 87–132, 3 pls.

[9] Andries, Maria. "Zur Systematik, Biologie und Entwicklung von Microdon Meigen." *Zs. Wiss. Zool.*, **103**, 1912, p. 300–361, 3 pls., 23 text-figs.

[10] Brauer, F. "Die Zweiflügler des Kaiserlichen Museums zu Wien. III Systematische Studien auf Grundlage der Dipteren-Larven, etc." *Denkschr. K. Akad. Wiss. Wien. Math. naturw. Cl.*, **47**, p. 1–100, 5 pls.

[11] Silvestri, F. "Contribuzioni alla Conoscenza dei Termitidi e Termitofili dell' Africa occidentale. II Termitofili. Parte seconda." *Boll. Lab. Zool. Gen. Agrar. Portici*, **14**, 1920, p. 265–319, 32 figs.

THE RÔLE OF ADRENAL SECRETION IN THE CHEMICAL CONTROL OF BODY TEMPERATURE

BY W. B. CANNON AND A. QUERIDO

LABORATORIES OF PHYSIOLOGY IN THE HARVARD MEDICAL SCHOOL

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Earlier work from these Laboratories by Aub, Bright and Forman,[1] and by McIver and Bright,[2] has shown that secretion from the adrenal medulla is capable of influencing the rate of metabolism in the body, and that increased secretion accelerates the oxidative processes. The erection of hairs, the ruffling of feathers, the constriction of peripheral vessels, the increase of blood sugar are well known phenomena occurring when warm-blooded animals are exposed to cold. They are signs that the sympathetic division of the autonomic system is in action. Is secretion from the adrenal medulla, which is admitted to be under sympathetic control, augmented when cold causes a discharge of sympathetic impulses?

In 1923 Hartman and his co-workers,[3] using the denervated iris as an indicator, observed dilation of the pupil when the animals (cats) were wet with cold water, or when wet with warm water and later cooled by a fan. The dilation did not occur after adrenal secretion was suppressed by removal of the glands or by removal of one gland and denervation of the other. The method has been severely criticised by Stewart and Rogoff.[4] Moreover, wetting the animals had emotional effects which might have simulated a reaction to cold. We have investigated the reaction, therefore, by another method.

We have employed as an indicator of adrenal secretion the denervated heart, in animals surviving and living normally. The heart is then influenced only by temperature changes and by agents brought to it in the blood stream. It is highly sensitive to circulating adrenin—0.00068 mg. of adrenalin per kilo per minute, injected intravenously, has increased the pulse 34 beats per minute. Cooling would decrease the rate. If cold makes the heart beat faster, therefore, the result would be in opposition to the influence of cold. The heart rate was determined by auscultation, by palpation and by electrical registration.

Animals thus prepared showed accelerations of heart rate varying from 12 to 43 per cent when taken to a cold room, accelerations from 34 to 43 per cent when placed in an ice box, an acceleration of 43 per cent when held in the lap and exposed to a cold draught from an open window, and accelerations between 27 and 64 per cent when cold water was introduced into the stomach. If the adrenal glands were rendered inactive the effects did not occur.

The use of cold water to produce what we have called a "heat liability"

has several advantages: it can be used anywhere, it is easily available at any time, and it is satisfactorily quantitative. The weight, the temperature and the specific heat of the water are known in relation to the weight, the temperature and the specific heat of the body, and thus the heat liability can be nicely adjusted to the purposes of the experiment.

If the heat liability is more than 1000 calories per kilo, acceleration of the denervated heart (indicating increased adrenal secretion) is almost uniformly attended by shivering. Thus two calorigenic factors, adrenin and muscular movements, are at work to protect the body against a fall of body temperature. If a heat liability of 900 calories is to be met, shivering rarely occurs, and if it occurs, it is of short duration (3 minutes). The heart rate is faster, however, showing that the adrenal factor is operating. If now the adrenal glands are rendered inactive, and a heat liability of 900 calories is established, shivering almost uniformly occurs and may last for as long as 17 minutes. Thus when the heat-producing service of the adrenal medulla is lacking, the shivering mechanism is resorted to.

Establishing a heat liability of 574 calories per kilo in man, by giving 750 cc. of water at 1°C, has caused the metabolism to increase as much as 25 per cent. The peak of the increase occurred in five different experiments from 12 to 25 minutes after the cold water was given. In no case did shivering appear. When 750 cc. of water at 34.5° was given the maximal increase was 4 per cent.

The foregoing experiments have an important bearing on the long-waged controversy over the question of true chemical augmentation of metabolism, apart from muscular activity. They support the contentions of Voit and of Rubner that such a mechanism exists and they account for the mode of action of that mechanism.

[1] Aub, Bright and Forman, *Amer. J. Physiol.*, **61**, (349) 1922.

[2] McIver and Bright, *Ibid.*, **68**, (622) 1924.

[3] Hartman, McCordock and Loder, *Ibid.*, **64**, (19) 1923; and Hartman and Hartman, *Ibid.*, **65**, (612) 1923.

[4] Stewart and Rogoff, *Ibid.*, **66**, (260) 1923.

INCREASE OF ADRENAL SECRETION IN FEVER

BY W. B. CANNON AND J. R. PEREIRA

LABORATORIES OF PHYSIOLOGY OF THE HARVARD MEDICAL SCHOOL

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In a previous communication,[1] Cannon and Querido have reported evidence that when warm-blooded animals are exposed to the danger of too great heat loss, secretion from the adrenal medulla is increased, and that this constitutes a true chemical calorigenesis. It has long seemed probable that the chemical control of heat regulation is involved in fever. The experiments now reported were undertaken to learn whether fever is associated with more abundant adrenal secretion. As in the earlier work, we have used the denervated heart as an indicator; the animals had survived the operation and were living normally in the laboratory.

As a routine means of producing a temporary fever we have used a suspension of dead typhoid bacilli, for the preparation of which we are indebted to Dr. Hans Zinsser. We injected 0.5 cc. of the suspension either intravenously or intramuscularly. We have also produced fever by intravenous injection of salt solution in tap water and by subcutaneous injection of adrenalin.

After injection of dead typhoid bacilli cats with intact adrenals have a characteristic course of fever temperature. There is first a rapid rise, reaching a maximum in about 34 minutes (average of 10 cases) and amounting on the average to almost 1°C. (0.93°C in 10 cases). Erection of the hairs and shivering commonly occur in this stage. Thereafter the temperature usually falls somewhat, but soon begins to rise again, but slowly and gradually, until it may reach a higher level than in the first elevation, and then slowly and gradually falls again. Shivering is not seen in this stage.

If the adrenal glands have been rendered inactive by removal of one and denervation of the other, injection of dead typhoid bacilli causes only the second slow rise, which, however, is in some instances only slight (e.g., 0.2° and 0.3°C).

Control observations to determine the normal variation of temperature during the day have shown a range of 0.3—0.4° C, whether the adrenals were intact or not.

The increase of rate of the denervated heart during the first elevation of temperature, in a series of tests on animals with adrenals intact, ranged from 30 to 78 beats per minute. The maximal rise of temperature in this series was 1.3°C. On the basis of Knowlton and Starling's study of the effect of temperature change on the isolated cat heart a rise of 1.3° in the region of 39°C would account at most for an increase of 15 beats per

minute.[2] Actual infection likewise has produced an increase of heart rate (24 beats per minute) not explained by the higher temperature (1.2°C). In an animal with adrenals inactivated intramuscular injection of 0.5 cc. of the dead typhoid bacilli caused an increase of 12 beats per minute during the second rise of temperature (2.2°C). That the heart rate was much faster in the injected animals when the adrenals were intact than could be accounted for by the higher temperature, and that this faster rate was absent in the animal with adrenals inactivated indicate that in fever the adrenal glands are stimulated to extra activity.

The erection of hairs, mentioned above, the increased blood sugar of fever, and increased adrenal secretion are changes which are seen also when the chemical mechanism for heat regulation is activated by cold.

Previous observers have reported reduction of adrenin in the adrenal medulla in consequence of infection.[3] Removal of the adrenals renders animals more susceptible to bacterial toxins.[4] It may be that the response of the adrenal medulla during toxic fever is serviceable to the organism.

[1] Cannon and Querido, *Proc. Nat. Acad. Sci.*, **10**, 245 (1924).
[2] Knowlton and Starling, *J. Physiol.*, **44**, 206 (1912).
[3] Elliott, *Ibid.*, **44**, 374 (1912); Borberg, *Skand. Arch. Physiol.*, **28**, 157 (1912).
[4] Lewis, *Am. Journ. Physiol.*, **64**, 503 (1923); Scott, *J. Exper. Med.*, **38**, 543 (1923).

AN EXPLANATION OF THE GAPS IN THE DISTRIBUTION OF THE ASTEROIDS ACCORDING TO THEIR PERIODS OF REVOLUTION

BY ERNEST W. BROWN

DEPARTMENTS OF MATHEMATICS AND ASTRONOMY, YALE UNIVERSITY

Read before the Academy, April 29, 1924

In 1866, Kirkwood, examining the elements of the orbits of asteroids known at that time, pointed out that they appeared to be distributed in such a manner that there were none which had periods in exact or nearly exact integral ratio with that of Jupiter. Since his time, the number available for statistical study has reached nearly four figures and with greater accuracy in the determination of their orbits, this feature has become increasingly evident. For example, to take one region, I found[1] in 1911, 8 with periods lying between 2100 and 2120 days, 5 with periods between 2222 and 2247 days, but none between 2121 and 2220 days, the last range including the half period of Jupiter, namely 2166 days. Just outside these limits the numbers become large. A similar statement can be made with respect to the region round a period one-third that of Jupiter and to a less extent in the regions with periods near $^2/_5$ and $^3/_7$ that of Jupiter.

It is true that these periods are in general those for the osculating ellipses and that they are therefore subject to change and further, that many of the orbits have poorly determined elements. But the former of these statements brings out more clearly the character of the phenomenon. We have to explain not only why no asteroid is certainly known with a period in integral ratio with that of Jupiter when the orbit is averaged over long intervals of time but also when we take an ellipse which represents the motion at any instant.

That resonance effects are not necessarily unstable in celestial motions is evidenced by the fact that certain well-known cases exist. The exact coincidence of the periods of rotation and revolution of the moon, and the six asteroids of the Trojan group where the ratio is 1 : 1 are the most striking cases, and others exist among the motions of the satellites.

The natural explanation, that the gaps in the distribution of the asteroids are due to the instability of the motions in those regions, has frequently been assumed as a fact, but in all the attempts hitherto made to prove it, no gravitational evidence of instability has appeared. In vol. IV, chapter XXV of his Mécanique Céleste, Tisserand gives two such attempts, with references to others. One of his investigations is defective because he assumes that the osculating eccentricity e at any time can be developed in powers of $(e - e_0)/e_0$, where e_0 is the mean value of e. This is certainly not the case when e_0 is small, as I proved in the paper referred to below; in any case, we cannot stop at the first or second power of this ratio. The method fails in the case of certain of the divisions of the rings of Saturn, supposedly due to the perturbations of its satellites; this in a mathematical sense, is the same problem.

The essential difference between these resonance problems and the usual methods of treating perturbations consists in the fact that the former cannot always be investigated properly by the method of *small* oscillations. It is for this reason that the work of Goldsborough[2] on the influence of the satellites on Saturn's ring fails to prove the instability of particles in the resonance regions, as Greaves[3] has suspected. On the general grounds of our knowledge of the problems of celestial mechanics a proof depending on small oscillations is not sufficient, as Poincaré[4] saw when he showed that no periodic orbit exists in the case where the resonance ratio is $i : 1 + i$, i being an integer, but did not draw the conclusion that actual orbits with this ratio were unstable in a physical sense. A simple example can be furnished by the case of a marble in a bowl with a small central bulge at the bottom. The marble can be balanced at the top of the bulge where it is unstable for infinitely small deviations. But it can make oscillations of finite amplitude over the surface of the bulge and the side of the bowl: from the physical point of view we get stable motions.

In a paper published[5] in 1912, I made an attempt to solve the problem

but only succeeded in showing that, for the case of the ratio 1: 2, the eccentricities must have a wide range reaching at least 0.2 during the motion, with a corresponding range of smaller amplitude in the period. The method adopted in the investigation of which this is a brief account is the same as that of the earlier paper, but is freed from certain of the limitations previously made. The main defect in these earlier papers is the neglect of the terms of the second order with respect to the eccentricities in the disturbing function and therefore the neglect of terms of the first order in the equations for $e\cos\alpha$, $e\sin\bar{\omega}$. It is the inclusion of these terms, which are shown to be necessary, that has made the change in the results. From the point of view of the theory of linear differential equations, it was previously assumed that the complementary function was periodic and therefore that one only needed to find the particular integral. I prove that the former is not periodic when there is resonance.

The chief difference in the method of treatment consists in the fact that the equations are put into such a form that not only can the essential characteristics of the motion be clearly seen but the equations themselves can be integrated. In fact, the first approximation is obtained by separating the dependent variables involved, and there appears to be no bar to its analytical combination to higher approximations. In a strictly logical sense, it cannot be regarded as a "proof." But since the instability which is demonstrated must be developed in a finite interval of time which is of the same order as that of the periods of revolution of the perihelia in ordinary cases (less than 10,000 years), there can be, I believe, little doubt that it furnishes the explanation for the gaps in the distribution of the asteroids and for the divisions in the rings of Saturn. While the cases $i: i+1$ and $i: i+2$ are treated in the memoir, I shall for the sake of simplicity give here only the fundamental equations for the case 1: 2, generally known as the Hecuba group.

Let n be the osculating mean motion of the asteroid, n' that of Jupiter and put $n = 2n'(1+x)$. In cases of approximate resonance x is small with a small mean value. When the resonance is perfect, x is an oscillating function with a mean value nearly zero. More accurately, if w for the osculating mean longitude of the asteroid and w' that of Jupiter, the derivative of $w - 2w' = \tau$ has a mean value equal to zero. This is the essential condition for exact resonance and the variations of τ, known as librations, are oscillatory in character when the motion is stable.

I first show that when we neglect the squares and higher powers of the eccentricities, which is permissible when the latter are not too large, the equation for x can be put into the form

$$\frac{1}{4n'^2}\frac{d^2x}{dt^2} = -\frac{1}{2}x(x^2 - c) + p, \qquad (1)$$

where p is a certain coefficient in the disturbing function depending only on the mass of Jupiter and the mean value of the ratio of the major axes of the asteroid and Jupiter and c is an arbitrary constant differing for different asteroids; p can be taken to be the same for all asteroids within or very close to the libration region.

For these cases it is shown that all the terms in this equation must be of the same order of magnitude; so that no periodic solution in which the range of x is small compared with its mean value exists, in conformity with Poincaré's theorem mentioned above. But there may be periodic solutions of this equation in which this range is finite. It is, however, not necessary to prove the existence of such solutions, as will appear immediately.

The equations for the eccentricity and longitude of perihelion are found in the form

$$\left.\begin{aligned} \frac{1}{2n'}\frac{ds}{dt} - s(R_2 - R_{20}\cos 2\tau) + c\,R_{20}\sin 2\tau &= S, \\ \frac{1}{2n'}\frac{dc}{dt} + c(R_2 + R_{20}\cos 2\tau) - s\,R_{20}\sin 2\tau &= C. \end{aligned}\right\} \qquad (2)$$

Here $s = e\sin\bar{\omega}$, $c = e\cos$; $\bar{\omega}\,R_2$ is proportional to the coefficient of e^2 and R_{20} to that of $e^2\cos(2\tau + \bar{\omega})$ in the ordinary development of the disturbing function and have the same properties as p; while S, C are periodic functions of τ, with constant terms present. Assuming that a solution of (1) has been obtained in a periodic form, S, C may be regarded as periodic functions of t, and in any case as known functions of t.

Let us first consider the ordinary planetary theory. In order to obtain the so-called "secular" terms, we neglect the periodic terms in the disturbing function so that in equations (2) we put $R_{20} = 0$, $S = \text{const.}$, $C = \text{const.}$, the perturbations of the orbit of Jupiter being neglected. The equations (2) then furnish the solution

$$\begin{aligned} e\sin\bar{\omega} &= e_0\sin(2n'R_2t + \bar{\omega}_0) + \text{const.}, \\ e\cos\bar{\omega} &= e_0\cos(2n'R_2t + \bar{\omega}_0) + \text{const.}, \end{aligned}$$

where e_0, $\bar{\omega}_0$ are arbitrary constants. This procedure is available as long as the angle τ makes complete revolutions and as long as the mean period of revolution of τ is short compared with $2\pi/2n'R_2$. It is the method used in finding the orbits of the major planets. For purposes of convenience the sine and cosine of the angle in the right hand members are developed in powers of the time and the results are called the secular changes of the eccentricity and perihelion. The portions of s, c depending on the periodic terms can then be found and the results give series which are available for numerical calculation.

In cases of libration either τ oscillates about some mean value τ_0 or (1)

has no periodic solution. In the latter case instability is proved. In the former case, let us put $\tau = \tau_0 + \tau_1$ so that the mean value of τ_1 is zero. As a first step let us neglect τ_1. The equations (2) then become linear with constant coefficients in the left hand members. To obtain the "complementary function" we put $S = 0$, $C = 0$ and try the solution $s = A \exp. 2qn't$, $c = B \exp. 2qn't$. After the elimination of the ratio A/B, the equation for q proves to be

$$q^2 = R_{20}^2 - R_2^2,$$

τ_0 having disappeared. If $|R_{20}| > |R_2|$, the value of q is real and $e \cos, \bar{\omega}$ $e \sin \bar{\omega}$ are expressed by exponential instead of by periodic functions. It is easy to show that for the case of the resonance ratio 2:1, R_{20} is nearly equal to $8R_2$. *In this case therefore the eccentricities are unstable and the motion if once set up cannot continue to exist in the assumed form.*

For the case of the Trojan group where the ratio is 1:1, the equation (1) has a periodic solution the amplitude of which may be indefinitely small and $R_{20} = 0$. There is therefore, as far as this part of the investigation is concerned, no apparent instability.

Although this is an approximation in so far as the lowest powers of the eccentricities are alone retained in the equations (1), (2), the character of the proof indicates that the result will not be fundamentally changed by the inclusion of their higher powers, as long as the eccentricity is not too large, especially in view of the wide margin between the values of R_{20}, R_2. It can also be shown that when τ_1 is included, its presence will not change the sign of q^2. The essential point of the proof is that one of the functions $\cos 2\tau$, $\sin 2\tau$ will have, in libration cases, a mean value which is not zero: in non-resonance cases the mean values of both these functions are zero.

When τ makes complete revolutions in a period which is long compared with $2\pi/2n'R_2$, the treatment is somewhat different, but the same result appears. Thus, as might be expected, asteroids very close to the libration region, as well as those within it, are unstable. Since the limits of the libration region are shown to be of the same order of magnitude as those observed, the wide gaps observed in the distribution of the periods of the asteroids is accounted for.

There is further significance in the forms of the equation (1), (2). Taken alone, the equation (1) giving the perturbations of the period or (what is the same thing) of the major axis, may possibly possess a periodic solution of finite amplitude; it is the equations for the perturbations of the eccentricity which exhibit the instability. This separation is of course a mathematical device, but it has a physical bearing. It is always noticeable in treating problems of celestial mechanics that the major axes are remarkably stable, that is, to a high order of approximation their oscillations are

periodic (cf. Poisson's theorem). In the theory of the Trojan group, it was shown[6] that the only direction in which instability was threatened appeared in the eccentricities, being due to a perturbation caused by Saturn. I was not able to obtain any definite result owing to the high order of the approximation needed. But these results indicate that one should search for instabilities, not primarily in the equation for the mean distances, but rather in those for the eccentricities.

There is no information at present as to what happens when the eccentricities are large. We can then no longer assume that their squares and higher powers may be neglected in a first approximation. But there is a challenge in the distribution of the short period comets. In the list given by Professor H. N. Russell,[7] there are nine comets with periods between 5.42 and 5.90 years, 15 with periods between 6.35 and 6.70 years and none with periods between 5.90 and 6.35 years. The half period of Jupiter is 5.93 years, so that for observational reasons instability due to resonance is not unlikely.

[1] *Science*, **33**, 86 (1911).
[2] *Phil. Trans. R. S.*, **222A**, 101–130 (1921).
[3] *Mon. Not. R. A. S.*, **82**, 356–360 (1922).
[4] *Les Nouv. Méth. de la Méc. Cél*, **1**, 101.
[5] *Mon. Not. R. A. S.*, **72**, 609–630 (1912).
[6] *Astron. J.*, **35**, 78 (1923).
[7] *Astron. J.*, **33**, 52 (1920).

NOTE ON THE DIFFERENCE IN VELOCITY BETWEEN ABSOLUTELY BRIGHT AND FAINT STARS

BY J. H. OORT

YALE UNIVERSITY OBSERVATORY, NEW HAVEN

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It is a well known fact that the average velocity of the so-called "dwarf" stars is considerably higher than that of the "giants" of the same spectral class. The following note is intended to show that the stars of high velocity (which are distinguished from the other stars by the peculiarly systematic character of their motions) are responsible for the major part of this difference, greater percentages of these stars occurring among the fainter stars than among the brighter ones.

The material used for the comparison consists of all F, G, K and M stars with Mt. Wilson spectroscopic parallaxes for which radial velocities have been published; for these stars I computed the transverse velocity 4.74 μ/π and corrected the radial velocities for the solar motion (assumed to be 20 km.).

As the stars with velocities smaller than about 65 km. show very little of the characteristics of the high velocity stars, it may be expected that in excluding all total motions higher than 65 km. we should get rid of the great majority of this class of systematically moving stars. In computing the total motion for this purpose the transverse velocity was not corrected for solar motion. This course was followed for convenience, but it has another advantage: as the apex of the sun's motion nearly coincides with the center of the hemisphere toward which the high velocity stars are moving this procedure will exclude more completely the stars moving in that general direction.

In order to effect the comparison between stars of different absolute magnitude, they were divided into strictly homogeneous groups according to the value of μ/π, and the comparison of the average radial velocities was made for each group independently. For the groups with high transverse velocities the comparison would have little meaning because all total velocities higher than 65 km. had been excluded. Accordingly only stars of transverse velocity less than 45 km. have been used.

In this preliminary note only the differences between "giants" and "dwarfs" considered as a class will be given, the division-point being taken at $+2^{M}.5$ for the F-type and at $+3^{M}.0$ for the other types. It looks as though there were a conspicuous increase in velocity for the K stars from -1.0 to $+2.0$ spectroscopic magnitude, but the average for the "giants" is not perceptibly smaller than that for the "dwarfs." The G-type "giants" do *not* show a distinct progression of velocity with absolute magnitude and for the M stars the progression is in a direction opposite that for the K stars.

The ratios of the average radial velocity of the "dwarfs" to that of the "giants" are given in the following table. Besides the division into homogeneous groups with respect to transverse velocity the computation has been made separately for three areas, area I comprising all stars more than 70° from either stream vertex, area II those between 45° and 70° from both, and area III those less than 45° distant. The weights used to combine the average velocities from the different areas and spectra have been taken in inverse proportion to the squares of these average velocities. The last column in the table gives the ratios found by Adams, Strömberg and Joy.[1]

SPECTRUM	RATIO OF AVERAGE VELOCITIES AREA I	AREA II	AREA III	WEIGHTED MEAN	PROBABLE ERROR	RATIO FROM MT. WILSON CONTR. 210
FO to *Fg*	0.92	0.87	1.30	0.99	±0.09	1.33
GO to *Gg*	0.81	1.25	1.65	1.18	±0.09	1.48
KO to *Kg*	1.12	0.94	1.24	1.08	±0.10	1.48
M	..	..	..	0.70	±0.19	1.77

The average ratio for all types comes out *1.04 ± 0.05* (p. e.), indicating an increase of about 0.12 ± 0.15 km. per magnitude, whereas the corresponding increase found in Mt. Wilson Contributions No. 210 is about 1.5 km. The value found here rests on a comparison of 305 "giants" with 222 "dwarfs."

The difference between the two ratios is perhaps most pronounced among the M stars. Because of the small number of "dwarfs" available I included stars with transverse velocities up to 50 km. in this type. After excluding the high velocities 11 "dwarfs" remain for the comparison showing an average radial velocity of 14 km. These stars are nearly 10 magnitudes fainter than the "giants" in this class having an average velocity of 18 km. In comparing the results for the three areas it appears that the ratio of the velocities is larger in area III than in the other areas, the average being 0.93 for area I, 0.98 for area II, and 1.39 for area III. The difference is probably real and indicates a stronger preferential motion among the "dwarf" stars, conforming with the results found by B. Boss[2] and Strömberg.[3]

Every exclusion of high velocities tends to diminish differences in average velocity previously found; however, in the case of an approximately Gaussian distribution and a limit of exclusion of the radial velocities at 59 km. (the average limit in the present comparison) the decrease is easily found to be small. That not all differences in velocity have disappeared is also illustrated by the fact that stars near the stream vertices and far from them still differ in their radial velocities. The ratio of the average radial velocity in area III to that in area I is found to be 1.44, which is only slightly smaller than the ratio we should expect without excluding high velocities. We can effect the same general diminution of the factors without the exclusion of any star on account of its high radial velocity, by repeating the comparison for stars moving toward the hemisphere from which the high velocity stars seem to come. A preliminary computation gave practically the same results as were found above.

The result of the present investigation is, therefore, that for stars with total velocities less than 65 km., or also for stars moving in a direction opposite to that in which the motions of the high velocity stars occur, the average velocities of "giant" and "dwarf" stars are nearly equal, the weighted mean ratio found being 1.04 ± 0.05 (p. e.). There is an indication, however, that the preferential motion is stronger for the "dwarfs" than for the "giants."

Neither the low velocity of the M-"dwarfs," nor the equality of the velocities of "giant" and "dwarf" stars for the later types,[4] nor any of the indications obtained from a comparison of double and single stars,[5] seem to support the hypothesis of a general increase of velocity with decreasing mass.

[1] *Mt. Wilson Contr.*, **10**, 1921 (181–198), from the corrected radial velocities in table VI.

[2] *Astron. J.*, **31**, 1918 (130–131).

[3] *Mt. Wilson Contr.*, **11**, 1922 (322).

[4] Russell, Adams and Joy, *Publ. Astr. Soc. Pacific*, **35**, 1923 (193), give values for the mean masses that would make the "giants" nearly twice as massive as the "dwarfs" of these spectral types.

[5] *Astron. J.*, **35**, 1923 (141–144).

ON A POSSIBLE RELATION BETWEEN GLOBULAR CLUSTERS AND STARS OF HIGH VELOCITY

BY J. H. OORT

YALE UNIVERSITY OBSERVATORY, NEW HAVEN

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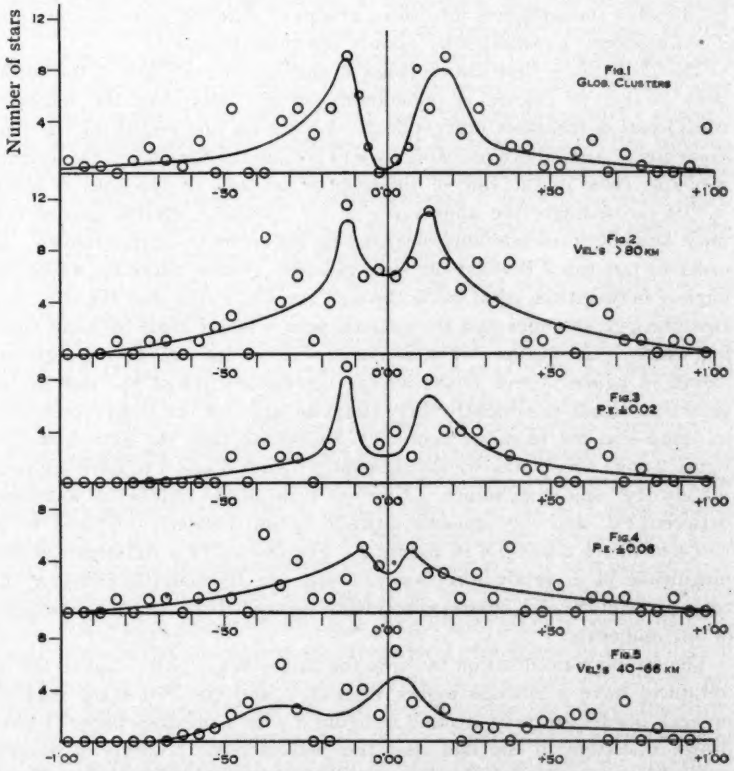
Several facts are known that might suggest a connection between globular clusters and stars of high velocity. The ten clusters for which radial velocities have been published by Slipher[1] give an average velocity of 140 km., well comparable with the higher star-velocities. The motions of the high velocity stars are known to be directed toward one hemisphere of the sky,[2] the center of which lies in the galactic equator at about 230° longitude; the directions of the radial velocities of these ten clusters lie between 123° and 334° galactic longitude and therefore fall practically within the same hemisphere.[3] Strömberg has remarked that a recent computation by Lundmark, with the aid of seven unpublished radial velocities gives a systematic motion of the clusters which nearly coincides with that of the stars of highest velocity.[4] It may be of significance that the mean antapex of the high velocity stars has been found to shift with decreasing average velocity towards lower galactic longitude[5]; that is, in the direction of the hemisphere in which nearly all the globular clusters are situated.

It has also been shown that the short-period Cepheids which are the typical variables in globular clusters generally have high velocities with the same characteristics as the other high velocity stars.

One of the most striking peculiarities of the globular clusters is their general avoidance of mid-galactic regions,[5,11] only one globular cluster being found within 5° of the galactic equator. Figure 1 shows the distribution according to the sine of the galactic latitude for the 88 clusters known to be globular.[6] Part of this avoidance may be explained by the existence of dark absorbing matter in the Milky Way[7]; Shapley however, has pointed out that there are several indications which favor the hypothe-

sis that globular clusters actually do not exist near the central plane of the galaxy (the minimum distance from this plane observed being 1200 parsecs).

It seemed of interest to inquire whether the total motions of the high velocity stars show an analogous avoidance, which might exist if these stars were in some way related to the system of the globular clusters. Dr. Strömberg very kindly sent me a list of 89 stars whose total velocities after the solar motion had been subtracted were higher than 100 km. I recom-



Comparison of the galactic distribution of globular clusters with that of the antapices of the motions of high velocity stars

puted the total motions of the 82 stars for which parallaxes, radial velocities and proper motions had been published. Mt. Wilson spectroscopic parallaxes[8] were used in this computation if smaller than 0."050; for the nearer stars I used trigonometric determinations. In addition a provisional

computation of lower velocities, from 40 to 80 km. was made in which only radial velocities published before 1923 were used. After correcting for solar motion the galactic latitudes of the antapices were computed. The results for 126 total velocities higher than 80 km. are shown in figure 2, giving the number of antapices for intervals of 0.05 in the sine of the latitude. Beyond $\pm 6°$ galactic latitude the antapices show a strong increase in frequency towards the galaxy, but between $+6°$ and $-6°$, where a maximum might have been expected, is a decided shortage of antapices.

That this phenomenon is not due to a local group of velocities is shown by dividing the antapices into three groups of different galactic longitude. The avoidance is exhibited by each of the three groups.

The probability that the shortage of motions in the plane of the Milky Way is due to chance is presumably small. Moreover the following consideration furnishes independent evidence for the reality of the phenomenon. Assuming a probable error of 20% in the parallaxes, the average probable error of the sine of the galactic latitude of the apex is about ± 0.04; accordingly the appearance of an avoidance of this narrow belt must have been considerably obscured by the errors in the parallaxes. In order to test this I divided the stars into two groups, those for which the angle χ between the great circle through the star's apex and the star, and that through the apex and the galactic pole is larger than 45°, and those for which it is smaller. The frequency-curves for the two groups are shown in figures 3 and 4; the average probable error of the sine of the galactic latitude is evidently three times as large for the stars represented in figure 4 as for those in figure 3. Supposing that the gap shown by figure 2 were accidental, we should expect figures 3 and 4 to have approximately the same appearance. Now the ratio of the number of antapices between $+6°$ and $-6°$ galactic latitude to that between $\pm 6°$ and $\pm 12°$ is *0.4* in figure 3 and *1.6* in figure 4. The chance of a difference of this magnitude in a certain direction is about one thousandth, bringing the total probability of a chance coincidence somewhere in the neighborhood of one millionth.

The strong concentration towards the Milky Way (only 7 out of the 65 antapices have a latitude higher than 30°) and the two sharp maxima on each border are very striking in figure 3. In a previous paper[3] I have drawn attention to the fact that the characteristic property of moving towards one hemisphere which is so strongly exhibited by all the stars with velocities higher than 66 km. disappears almost entirely below this limit. The Mt. Wilson radial velocities which were published after that article appeared agree well with the rather sudden change in the asymmetry of the velocities at this point. The distribution of the antapices of stars with total velocities between 40 and 66 km. does not disprove the existence of the limit: the curve in the last diagram which represents this distribution

for the stars with χ larger than 45° does not show any trace of a minimum in the Milky Way. On the other hand the stars with velocities between 80 and 100 km. still show the avoidance as far as the scantiness of material permits us to judge. We cannot make use of the velocities between 66 and 80 km. because of the large probable error existing in the lengths of the velocity-vectors. The question whether the secondary maximum at —20° latitude shown by figure 5 (and which is more or less visible in the other curves) is real must be postponed till a more complete investigation.

From the foregoing it appears likely that the avoidance of the galactic plane by the velocities higher than 80 km. is real and not an appearance due to local streams of stars. This peculiar characteristic would at first sight seem to point to a direct connection with the globular clusters; there are also other peculiarities in the motions of the high velocity star that would favor the hypothesis that these objects had actually escaped from the system of globular clusters. Our knowledge about the motions and the arrangement of the clusters, and especially about the velocities of these stars, is, however, so preliminary that it does not seem of much use to work out the consequences of such a hypothesis before more data about high total velocities have been collected.*

It may be of interest to mention briefly the other objects which show an avoidance of the Milky Way.

It is well known that the spiral nebulae are totally absent from the region near the galactic equator. A remarkable feature of these nebulae is that their belt of avoidance is more than twice as wide as that of the globular clusters: in the drawings published by Reynolds[9] no spiral is found within 12° of the galactic equator although with a random distribution we should expect to find about 80 nebulae below this latitude. Except perhaps for the stars constituting the clouds of the Milky Way the only other objects that seem to occur less frequently in the central parts of the galaxy than on its borders are the planetary nebulae. I counted the numbers of these nebulae in different galactic latitudes from the list published in Volume XIII of the Lick Publications. The frequencies show a very strong increase towards the lower latitudes except for a probably real and not local deficiency between +3° and —3° latitude, where only 12 nebulae are counted, whereas the number we should expect from the rest of the curve is about 35. Here the apparent zone of avoidance is much narrower than in the case of the clusters and can possibly be explained by absorbing clouds. But the fact that neither the most distant c-stars[10] nor the open clusters[11] show any trace of this absorption would then indicate a very great average distance (presumably over 2000 parsec) for both the required obstructing matter and the planetary nebulae.

* It may be remarked that the hypothesis of an extended ring of obstructing matter such as has been suggested to account for the absence of globular clusters from the central

parts of the Milky Way could hardly explain the shortage of high velocity-apices in those regions: in order to diminish the speed of a passing star perceptibly the mass of this ring would have to be of the order of 10^{20} stars or 10^{10} times the mass which Kapteyn attributes to the whole stellar system. According to the theory of relativity (De Sitter, *Mon. Not. R. Astron. Soc.* **77**, p. 176) a ring of this mass cannot exist at a distance of the order of that of globular clusters, as the resulting shift toward the violet in the spectra of globular clusters and spiral nebulae would be much greater than the observations permit.

[1] *Pop. Astron.*, **26**, 1918 (8).

[2] This was first observed by B. Boss, *Ann. Rep. Director Dep. Merid. Astr.*, Carnegie Inst. of Washington, 1918.

[3] Oort, *Bull. Astr. Inst. Netherlands*, **1**, 1922 (134).

[4] These PROCEEDINGS, **9**, 1923 (315).

[5] Shapley, *Mt. Wilson Contr.*, **8**, 1918 (134–139, 218–222, especially 314–319).

[6] *Mt. Wilson Contr.*, **8**, 1918 (314), *H. C. O. Bull.*, 776, 1922.

[7] Charlier, *Lund Meddel.*, 2nd Series, No. **19**, 1917 (Plate VII).

[8] *Mt. Wilson Contr.*, **9**, 1920 (423).

[9] *Mon. Not. R. A. S.*, **83**, 1923 (147–152).

[10] Schilt, *Bull. Astr. Inst. Netherlands*, **2**, 1924 (49).

[11] Shapley, These PROCEEDINGS, **5**, 1919 (344–351).

NOTE ON SOME STATISTICAL CONSEQUENCES OF THE LUMINOSITY LAW

BY WILLEM J. LUYTEN

HARVARD COLLEGE OBSERVATORY

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The luminosity curve derived by Kapteyn and van Rhijn is based on considerations of parallactic motions, and, as far as the part near and beyond the maximum is concerned, on the available material of numbers and parallaxes of stars with large proper motion. Although the rapid accumulation of the last two kinds of observational data will make a revision of the luminosity curve possible in the near future, it may be of interest to compare the statistical behavior of certain groups of selected stars with that to be expected from the luminosity law and some auxiliary assumptions, principally the velocity law.

For some parts of the sky our meridian and astrographic catalogues give complete lists of stars, brighter than a given apparent magnitude and with proper motions exceeding a given value. It is then an easy matter to derive the frequency function of $H = m + 5 + 5 \log \mu$ for these stars. A simple theoretical expression for this frequency curve $F(H)$ may be found in the following way.

The number of stars whose distances lie between r and $r + dr$, whose

absolute magnitudes are between M and $M + dM$, and whose linear velocities perpendicular to the line of sight lie between T and $T + dT$ is given in the usual notation by:

$$dN = 4\pi r^2 \Delta(r) dr . \Phi(M) dM . \psi(T) dT \qquad (1)$$

To find the total number of stars in the sky brighter than apparent magnitude m_c and with proper motions larger than μ_c we have to integrate the above expression between appropriate limits. In our applications we shall be dealing with stars of comparatively large proper motions, i.e. relatively near-by stars, and accordingly we may, in first approximation, assume that the density is constant throughout our space of integration. For $\Phi(M)$ we shall adopt Kapteyn's form, and for $\psi(T)$ Schwarzschild's, i.e. a normal error curve in log T with the arithmetic mean value of log T as a linear function of M. Substituting $u = 5 \log r$, $V = 5 \log T$, $g = 3/(5 \text{ mod},)$ and

$$V_0 = \alpha + \beta M \qquad (2)$$

equation (1) reduces to

$$dN = C . \exp\,[gu - h^2(M - M_0)^2 - t^2(V - V_0)^2] .\ dudMdV \qquad (3)$$

in which C, h, M_0, t, α, and β are constants of the distribution functions. N is then found by integrating (3) between the limits:
V from $A + u$ to $+\infty$, u from $-\infty$ to $B - M$, M from $-\infty$ to $+\infty$, where $A = 5 + 5 \log \mu_c$, and $B = m_c$.

However we want, not N, but $F(H)$ for these stars. Noting that $H = m + 5 + 5 \log \mu = M + V$, we effect the following transformation of variables: $M = (H + K)/2$, V $= (H - K)/2$, $u = u$. The restrictions resulting from the conditions that $m \rbrack m_c$ and $\mu > \mu_c$ now reduce to

$$A + u < (H - K)/2, \qquad u < B - (H + K)/2 \qquad (4)$$

giving a double restriction for u, and none for K. Written otherwise we have:

$$u < (H - K)/2 - A \quad (a) \qquad \text{and} \qquad u < B - (H + K)/2 \quad (b)$$

These two conditions are identical when $H = A + B = m_c + 5 + \log \mu_c = H_c$. After introducing H and K in (3) by means of the expressions given for M and V, and replacing $dudMdV$ by $^1/_2 . dudHdK$, $F(H)$ is given by:

$$F(H)dH = dH \int_{-\infty}^{+\infty} dK \int_{-\infty}^{(a)} f(H, K, u)du \quad \text{for } H < H_c$$

$$F(H)dH = dH \int_{-\infty}^{+\infty} dK \int_{-\infty}^{(b)} f(H, K, u)du \quad \text{for } H > H_c$$

$$f(H, K, u) = C . \exp\,[gu - h^2(H + K - 2M_0)^2/4 - t^2(H - K - 2\alpha - \beta H - \beta K)^2/4]$$

The limits (a) and (b) in the second one of both double integrals denote

that u has to be integrated between $-\infty$ and one of the limits indicated by equations (a) and (b).

The integrals are elementary, and it is easily seen that the solution may be written as follows:

$$\left.\begin{array}{ll} H < H_c & P(H)dH = C_\mu . i/\sqrt{\pi} . e^{-i^2(H-H_a)^2}dH \\ H > H_c & Q(H)dH = C_m . i/\sqrt{\pi} . e^{-i^2(H-H_b)^2}dH \end{array}\right\} \qquad (5)$$

$$i^2 = \frac{h^2t^2}{h^2 + t^2(1+\beta)^2}$$

$$H_a = H_o + g/2t^2 + g\beta(1+\beta)/2h^2$$

$$H_b = H_o - g(1+\beta)/2h^2$$

if H_o is the value of H derived from (2), and corresponding to the maximum M_o of the luminosity curve.

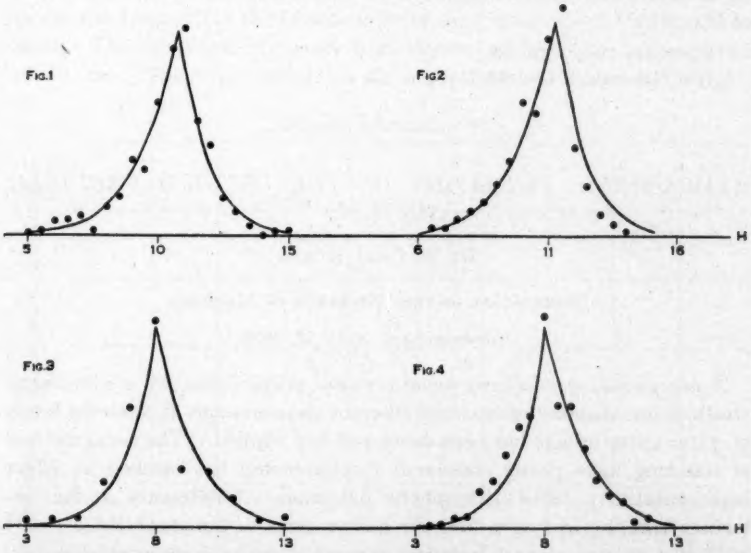
The result is remarkable in that P and Q are quite general, and except for C_μ and C_m which only convert the relative frequencies into absolute values, are independent of either μ_c or m_c. Equation (5) gives the solution for *all* values of μ_c and m_c, without need of computation of C_μ and C_m as explicit functions of these quantities. To obtain the result in any specific case we follow curve P up to $H = H_c$, and curve Q from thereon, multiplying the Q-frequencies by such a factor that for $H = H_c$ the P- and Q-frequencies are identical. Graphically the solution is given by keeping P constant, and raising or lowering Q (by multiplying its ordinates) until it intersects P at the desired point $H = H_c$. It is also significant that as long as we require only relative frequencies, the solution is the same for *all* values of μ_c and m_c which satisfy the relation $m_c + 5 \log \mu_c = \text{constant}$. It follows that the same relative curve must represent the frequency distribution of H for those stars whose apparent magnitudes and proper motions lie between such limits m_1, m_2 and μ_1, μ_2 that $m_1 + 5 \log \mu_1 = m_2 + 5 \log \mu_2 = H_c - 5$.

That the H-frequency curve must be "beaky" is immediately evident from the consideration that the stars used are brighter than a certain apparent magnitude and have a proper motion larger than a given value. Because the numbers of stars of *all* proper motions increase rapidly with decrease in apparent brightness, and likewise the numbers of stars of *all* magnitudes increase rapidly with decreasing proper motion the peak must appear, no matter what frequency laws are used.

Before comparing the theoretical results with the observations we must choose numerical values of our constants. For h and M_o we shall adopt Kapteyn's values,[1] 0.28 and 7.7, respectively; for t a value of 0.56 seems plausible; β we may take[2] as 0.28 and H_o as 12.5. We then have $i = 0.20_5$, $H_a = 17.9$, and $H_b = 1.4$. The resulting curve for $F(H)$ will therefore be symmetrical only when $H_c = H_s = 9.65 = (H_a + H_b)/2$; for

smaller values of H_c the ascending branch will be steeper; for larger values of H_c the descending branch of the curve will be the steeper.

In the Greenwich 1910 Catalogue (p. B XXXII) a list is given of all stars in the zone having proper motions larger than $0''.200$ per annum. This list is probably complete to about magnitude 9.2; H_c is then 10.7. The theoretical curve is shown in figure 1 as a full line, the dots represent the observed points from 224 stars. In figure 2 the comparison between



Comparison between theoretical and observed frequency curves F (H) for stars brighter than a given apparent magnitude and with proper motion exceeding a given value.

theory and observation for the case $H_c = 11.25$ is given (proper motions from the Greenwich Polar Zone, m_c is about 9.5 to 10.0 and $\mu_c = 0''.200$; 255 stars). Figures 3 and 4 show the case for $H_c = 8.0$; figure 3 representing the brighter stars, with $m_c = 5.5$ and $\mu_c = 0''.316$ (151 Boss stars), and figure 4 representing the fainter stars with $m_c = 7.0$, $\mu_c = 0''.158$ (209 stars from various zone catalogues). The only adjustment that has been made is the reduction of all curves, theoretical as well as observational, to the same total number of stars.

The agreement in figure 1 is as good as may be expected; the other three cases may still be deemed satisfactory, considering the fact that the "beaky" form of the curves makes them extremely sensitive to small changes in H_c and the constants used in the distribution curves.

On the other hand, the theoretical curves, exhibited here still depend

very largely on the absolutely brighter stars. A real test of the luminosity curve and the auxiliary laws can only be obtained if we use values of H_c at least as large as 14. Thus the Wolf proper-motion stars might be used to advantage were it not for the fact that the value of the systematic correction to be applied to Wolf's magnitudes is rather uncertain. There are indications, however, that with any reasonable value for this correction, the agreement between theoretical and observational curves will remain bad in this case, there being an observed excess of stars with large values of H.

[1] *Mt. Wilson Contr.*, No. 188.

[2] *Ann. Harvard Coll. Obs.*, **85**, No. 5, p. 103.

ATMOSPHERIC PULSATION OF THE CEPHEID VARIABLE, η AQUILAE[1]

BY W. CARL RUFUS

OBSERVATORY OF THE UNIVERSITY OF MICHIGAN

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A new method of analyzing variable radial velocity data by a systematic study of line displacements from different elements and at different levels of a star's atmosphere has been developed and applied. The usual method of selecting lines giving consistent displacements has masked an effect here tentatively called atmospheric pulsation. A difference in the velocities from the hydrogen and the helium lines in the star,[2] B.D. + 56° 2617, led to the idea of isolating the velocities of high-level elements. Application was made in the case of χ Aurigæ by isolating the hydrogen lines and a systematic distribution of the hydrogen velocity residuals with respect to phase was found.[3] Based upon the isolation of velocities at three levels in the star's atmosphere the method[4] adapted to the problem of Cepheid variation was first applied by Mr. J. A. Aldrich in a study of S. Sagittæ.[5] The method is also applicable to long-period and irregular variables, to secular and periodic changes in peculiar spectra, and to other astrophysical problems.

The Method as Applied in η Aquilae.—Velocity-difference curves correlated with the light variation and other periodic changes characterize the method here applied. The radial velocities of concentric layers of the star's atmosphere are determined separately by isolating certain elements, and by grouping lines of the spectrum originating at assumed levels based upon the determination of their heights in the sun by St. John[6] and by Mitchell.[7] The approach of two layers indicates compression of

the intervening gases and recession indicates expansion. Resulting changes in temperature, density, radiation, and absorption, may be correlated with the light variation and changes of spectrum to explain many of the anomalous characteristics of Cepheid variation. Inasmuch as spectroscopic work is limited to surface radiations characterized by atmospheric effects, a new method of analysis to determine changes of atmospheric conditions should yield results of fundamental significance.

The Velocity Curves.—Forty-two plates of η Aquilæ made with the single-prism spectrograph of the Observatory of the University of Michigan were used. The mean velocity curve from lines of all levels has an amplitude of 40 km. The chief feature is a "hump," not obtained by Wright,[8]

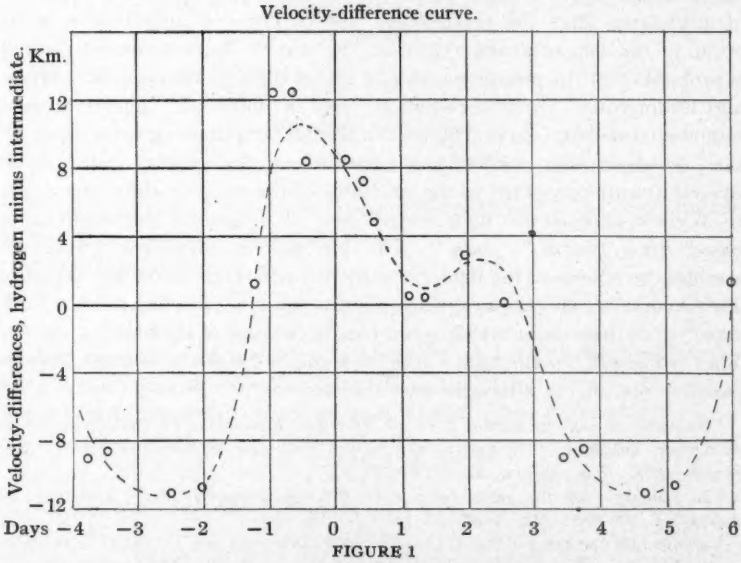

FIGURE 1

Positive values indicate compression and negative values indicate expansion of the atmospheric gases between the two effective levels.

which occurs about one day, (one-seventh of the period), after the "Still-stand" of the light curve by Wylie.[9] This irregularity resembles the typical "hump" of a Cepheid velocity curve, which Curtis represents quite satisfactorily in some cases by adding a single oscillation to an elliptical curve.[10] Velocity curves for high, intermediate, and low levels, and separate curves for hydrogen and strontium, were also formed. The secondary feature is more prominent in the high and the low level curves, and is greatly emphasized in the case of the isolated elements, but is lacking in the intermediate level. This feature of the velocity curves of η Aquilæ

occurs at the time of greatest radial expansion of the star as a whole, on the basis of the pulsation theory, with a lag of phase of the upper layer. The difference in form and phase of these curves indicates that the ordinary velocity curve of a Cepheid is not definite, and the "elements of an orbit" depend upon the selection of lines.

The Velocity-Difference Curves.—The differential method, automatically eliminating plate errors, may be expected to increase the degree of accuracy of these curves. High-minus-intermediate, intermediate-minus-low, and hydrogen-minus-intermediate curves were formed. Positive differences represent compression. In general, compression of the atmosphere occurs after maximum compression of the star as a whole, which accounts for the retardation of the light maximum in accordance with the suggestion of Eddington that the retardation, not determined analytically, must occur in the non-adiabatic region of the star.[11] With sufficient data it is probable that the pulsations may be traced through the successive layers and Eddington's "gusts of radiation" may be observed. The hydrogen-minus-intermediate curve (Figure 1) has an amplitude greater than 20 km., which exceeds one-half the amplitude of the velocity curve itself. Special attention is called to the similarity of this velocity-difference curve to Wylie's photometric light curve,[9] and to a spectral variation curve based upon Shapley's data.[12] The intermediate-minus-low curve resembles the reverse of the light curve with synchronous secondary features. *The "humps" of the velocity-difference curves are synchronous with the "Stillstand" of the light curve*, which is not true in the case of the velocity curves. The "Stillstand" of the light variation seems to be due to a stage of comparative rest in the atmosphere of the star.

[1] Presented in part to Section D of the American Association for the Advancement of Science, December 1923; and in part to the American Astronomical Society, December 1923. *Pop. Astron.*, **32**, 1924 (228).

[2] In January 1920 the author discovered this spectroscopic binary, announced in *Astroph. J.*, **51**, 1920 (252), and made the following record:

Variable radial velocity of star B.D. +56° 2617 discovered Jan. 14, 1920. The plates on hand show a range for the hydrogen lines from +92.6 to —65.2, nearly 160 km. The H and K lines appear to be nearly stationary, only 8 km. range in the set of five plates measured, with a displacement averaging —25 km. *The helium lines seem to indicate a much smaller range of displacement than the hydrogen.* Further investigation is desirable.

[3] Rufus, W. Carl, *J. Roy. Astron. Soc. Can.*, **14**, 1920 (139).

[4] Rufus, W. Carl, "Atmospheric Pulsation of Cepheids, A Method of Attack," presented to the American Astronomical Society, September 1923. *Pop Astron.*, **32**, 1924 (22).

[5] Aldrich, J. A., "A Study of S Sagittæ," a thesis under the direction of Professor R. H. Curtiss, to be published by the Observatory of the University of Michigan. The isolation of the velocities from the hydrogen lines was proposed to Mr. Aldrich by the writer and systematic motion with reference to the metallic elements was found. The isolation of the velocities from lines of intermediate level was then proposed to Mr.

Aldrich by Professor Curtiss. Mr. Aldrich's results were submitted to members of the faculty of the University of Michigan in May 1923 and were presented to the American Astronomical Society in December 1923. *Pop. Astron.*, **32**, 1924 (218).

[6] St. John, Charles E., *Astroph. J.*, **37**, 1913 (322) and **38**, 1913 (341).

[7] Mitchell, S. A., *Ibid.*, **38**, 1913 (407).

[8] Wright, W. H., *Ibid.*, **9**, 1899 (62).

[9] Wylie, C. C., *Ibid.*, **56**, 1922 (225).

[10] Curtis, Heber D., These PROCEEDINGS, **9**, 1923 (187).

[11] Eddington, A. S., *Mon. Not. Roy. Astron. Soc. London*, **79**, 1919 (181).

[12] Shapley, H., *Astroph. J.*, **44**, 1916 (287).

ON PARAMETRIC REPRESENTATIONS OF CONTINUOUS SURFACES

BY B. DE KERÉKJÁRTÓ

PRINCETON UNIVERSITY

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1° The following theorem on parametric representations of continuous curves has been proved by Fréchet:[1]

Given the functions

$$x = f(u),\ y = g(u),\ldots,\ z = h(u)$$

continuous and uniform in the interval $0 \leqq u \leqq 1$ which thus represent a continuous curve in euclidian space with the coördinates $x, y, \ldots, z$, it is possible to represent the curve by a change of the parameter

$$u = u(t),\ 0 \leqq t \leqq 1$$

in such a way that the functions

$$x = f(u(t)) = F(t),\ y = G(t),\ldots,\ z = H(t)$$

are not simultaneously constant in any interval of t.

Let us take indeed for

$$t = t(u)$$

a continuous uniform function which is monotonically increasing from $t(0) = 0$ to $t(1) = 1$ except in those intervals in which all the functions $f(u), g(u),\ldots, h(u)$ are simultaneously constant and in which the function $t(u)$ also should be constant.[1]

2° We want to investigate the analogous question for the case of continuous surfaces that is to say for continuous uniform images of surfaces, e.g. for those of the sphere.[2]

Let S be a sphere and

$$x = f(u, v),\ y = g(u, v),\ldots,\ z = h(u, v)$$

continuous uniform functions on the sphere S. Let C be a continuum on the sphere consisting of at least two points such that all the functions $f(u, v)$, $g(u, v),\ldots, h(u, v)$ are simultaneously constant on C and C is not contained in any larger continuum having the same property. Let us consider the set E of points belonging to the different continua C.

If every continuum C has for complementary set one single region the complementary set of E on the sphere will consist also of one single region G.[3] In this case it is possible to transform the region G by a continuous (1, 1)—transformation into a region G' (lying on another sphere S') whose boundary is a discontinuous point set.[4] If we denote by (s, t) the coördinates of the sphere S' and by

$$u = u(s, t),\ v = v(s, t)$$

the transformation between G and G' we obtain by elimination of (u, v) a representation of the given continuous surface

$$x = f(u(s, t), v(s, t)) = F(s, t),\ y = G(s, t),\ldots,\ z = H(s, t)$$

such that the functions $F(s, t)$, $G(s, t),\ldots, H(s, t)$ are not simultaneously constant on any continuum of the (s, t)-sphere consisting of more than one point.

3° Without the above condition upon the continua C the same proposition does not hold. Take for instance the functions

$$x = \cos u.\cos v,\ y = \tfrac{1}{2}\sin 2u.\cos^2 v,\ z = \tfrac{1}{2}\cos u.\sin 2v$$

$$(0 \leqq u < 2\pi,\ -\pi/2 \leqq v \leqq +\pi/2)$$

and consider the continuous surface

$$x^2(x^2 - 1) + y^2 + z^2 = 0$$

defined by means of them. This is divided by the origin $x = y = z = 0$ into two surfaces each of them being homoeomorphic with the sphere. On the circle $u = \pm \pi/2$ all the functions x, y, z are constant (namely equal to zero). For any other parametric representation of the same continuous surface as a continuous uniform image of the sphere the points of the sphere which correspond to the points $x > 0$ of the surface form one or several regions whose sum is a true subset of the sphere. Any continuum on the boundary of any such region is transformed into the single point $x = y = z = 0$ so that on such a continuum consisting of more than one point all the functions x, y, z are constant.

4° Let us consider again the set E as defined above, without the hypothesis that no continuum C subdivides the sphere. The complementary set of E consists of an *enumerable* infinity of regions by a well-known theorem.[5] Each of these regions can be transformed by a continuous (1, 1) transformation into a region on a sphere S^i whose boundary is a discontinuous point set. Every such sphere defines by its related transformation a continuous surface, the corresponding functions being constant on no continuum of the sphere S^i consisting of more than one point. The sum of these continuous surfaces (which are in enumerable infinite number) gives the original continuous surface.

5° Another "normal representation" of continuous surfaces which may be more satisfactory can be obtained as follows. Let us denote by $G_1, G_2, \ldots$ the complementary regions of the set E. For any two regions G_i and G_j there is at most one continuum C (of the set E) which has points on the boundaries of both of them. Hence it follows that those continua C which have points on the boundaries of more than one region G_i form an enumerable set $C^1, C^2, \ldots$.

We transform now the region G_1 into a region G_1' situated on another sphere S', bounded by a set of non-intersecting circles and by a discontinuous point set, in such a way that every continuum of the boundary of G_1 which contains at least one point of a continuum C^i corresponds to a circle, every other continuum of the boundary of G corresponds to a single point on the boundary of G_1'[6]. Let c^1 be a circle belonging to the boundary of G_1', and C^1 the corresponding continuum on the sphere S (whose subset is that continuum of the boundary of G_1 which corresponds to the circle c^1). Let us denote by $G_{\alpha_1}, G_{\alpha_2}, \ldots$ those regions G_i whose boundaries contain points of C^1.

If there is only a finite number of such regions we divide the interior of the circle c^1 (i.e. that region on the sphere which is bounded by c^1 and which does not contain G_1) into the same number of partial regions by meridian arcs passing through the pole of S' with regard to c^1 (we understand by poles of S' with regard to c^1 the extremities of the diameter of S' orthogonal to the plane of c^1). Let us assign to each of these regions one of the indices $\alpha_1, \alpha_2, \ldots$ so that every index α_i belongs to one and only one of the partial regions.

If there is an infinity of regions $G_{\alpha_1}, G_{\alpha_2}, \ldots$ whose boundaries contain points of C^1 we divide the interior of c^1 into an infinite number of partial regions in the following way: we take a meridian arc from the pole to a point of c^1 and an infinite sequence of such arcs which converge monotonically to the first one. Call g_k the region determined in the interior of c^1 by the k^{th} and $k+1^{\text{th}}$ arc which does not contain any other arcs. For any positive number δ there is only a finite number of regions G_{α_l} such that the value

$[f(u, v) - f(u_0, v_0)]^2 + [g(u, v) - g(u_0, v_0)]^2 + \ldots + [h(u, v) - h(u_0, v_0)]^2$ where (u, v) is a variable point of G_{α_l} and (u_0, v_0) a point of C^1 exceeds δ; we assign to the region g_l the index α_l.

We leave out of g_l a set of non-intersecting circular surfaces and a discontinuous point set so that the region G'_{α_l} so obtained is equivalent to G_{α_l}. We transform the region G_{α_l} into G'_{α_l}, by a continuous (1, 1) correspondence, so that the continuum of the boundary of G_{α_l} which belongs to C^1 shall correspond to the boundary of g_l, those continua of the boundary of G_{α_l} which contain points of the other continua C^i shall correspond to the circles on the boundary of G'_{α_l}; the other continua of the boundary of G_{α_l} shall correspond to the discontinuous point set on the boundary of G'_{α_l}.

On every boundary circle of the region $G'_1 + G'_{\alpha_1} + G'_{\alpha_2} + \ldots$ we repeat the same operation. For the indefinite continuation we only want to satisfy the following condition: if c^{β_2} is any circle inside of c^1 belonging to the boundary of a region G'_{α_l}, c^{β_3} a circle inside of c^{β_2} obtained in an analogous way, and so forth, the diameters of these circles converge to zero if and only if the limiting set[7] of the continua C^{β_2}, $C^{\beta_3}, \ldots$ does not contain points of a continuum C^i.—An enumerable infinity of repetitions of the above process (which will not necessarily be of the type ω) leads to an end.

By means of the transformations of the regions G_k into the regions G'_k we obtain on the new sphere S' continuous uniform functions corresponding to the functions $f, g, \ldots, h$ which are not constant simultaneously on any continuum of a region G'_k. The points on the sphere S' which do not belong to the regions G'_k (nor to the discontinuous point sets figuring in the boundaries of G'_k) form a nowhere dense closed set composed of an enumerable infinity of simple Jordan arcs.—Thus we have the following result:

Given a continuous surface (which is continuous uniform image of a sphere), by a continuous (1, 1) transformation of the parametric representation we can obtain a parametric representation of the same continuous surface such that the functions which define its coördinates are not simultaneously constant on any continuum of the sphere except a nowhere dense closed point set composed of an enumerable infinity of simple Jordan arcs.

6° It seems to me that this result cannot be improved if we consider *the continuous surface itself* and not merely *the point set furnished by it.* This idea will be understood by the following two remarks. First, the two continuous curves

1. $x = \cos 2\pi u,\ y = \sin 2\pi u \quad (0 \leqq u \leqq 1)$

and 2. $x = \cos 4\pi u,\ y = \sin 4\pi u \quad (0 \leqq u \leqq 1)$

are different regarded as continuous curves though they represent the same point set. Secondly, the continuous surface:

$$\left.\begin{aligned} x &= \sigma\,|\cos u.\sin 2v|, \\ y &= \sin 2u.\cos^2 v, \\ z &= 1 - 2\cos^2 u.\cos^2 v; \end{aligned}\right\} \quad \left\{\begin{aligned} &-\pi/2 \leqq u \leqq +3\pi/2,\ -\pi/2 \leqq v \leqq +\pi/2; \\ &\sigma = -1, \text{ if } -\pi/2 \leqq u \leqq +\pi/2, \\ &\sigma = +1, \text{ if } +\pi/2 \leqq u \leqq +3\pi/2; \end{aligned}\right.$$

is as point set coincident with the sphere

$$x^2 + y^2 + z^2 = 1$$

but as continuous surface it is of the type of the surface

$$x = \cos u.\cos v,\ y = {}^1/_2 \sin 2u.\cos^2 v,\ z = {}^1/_2 \cos u.\sin 2v$$

mentioned above (3).

[1] Fréchet, *Rend. Circ. Mat. Palermo*, **22** (1906), p. 67 et seq.

[2] The idea of such a problem occurred to Prof. Fréchet by writing a paper (on the "distance" of two surfaces) soon to appear in the "*Ann. Soc. Math Polonaise*". This question was mentioned to me by Prof. Fréchet in a letter of Feb. 20, 1924, to whom I wrote the solution, as given in the text, in a letter of March 5, 1924.

[3] Kerékjártó, *Vorlesungen über Topologie*, **1**, (Berlin, 1923), p. 49.

[4] Kerékjártó, *l. c.*, p. 48, 123.

[5] See, for instance, Kerékjártó, *l. c.*, p. 31.

[6] The possibility of such a representation of G_1 upon a region G_1' follows from the results of Kerékjártó, *l. c.*, p. 48, 123; comp. also p. 165 et seq.

[7] Kerékjártó, *l. c.*, p. 38.

THE WAVE-LENGTH OF MOLYBDENUM Kα RAYS WHEN SCATTERED BY LIGHT ELEMENTS

BY ARTHUR H. COMPTON AND Y. H. WOO

RYERSON PHYSICAL LABORATORY, UNIVERSITY OF CHICAGO

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A paper by Clark, Stifler and Duane in the April number of these PROCEEDINGS describes measurements of the wave-length of the X-rays from a molybdenum target after they have been scattered by certain substances of low atomic number. The conclusion drawn from these experiments is that no secondary radiation occurs whose wave-length is increased by the amount 0.024 $(1-\cos\theta)$ A. U. predicted by the quantum theory of scattering. They find, on the other hand, evidence for modified secondary radiation whose minimum wave-length is $\lambda\lambda_k/(\lambda_k-\lambda)$, where λ is the wave-length of the incident rays and λ_k is the critical K absorption wave-length of the radiating element. The experiments described in the present paper were undertaken to examine this question in greater detail.

The apparatus used was identical in general design with that employed

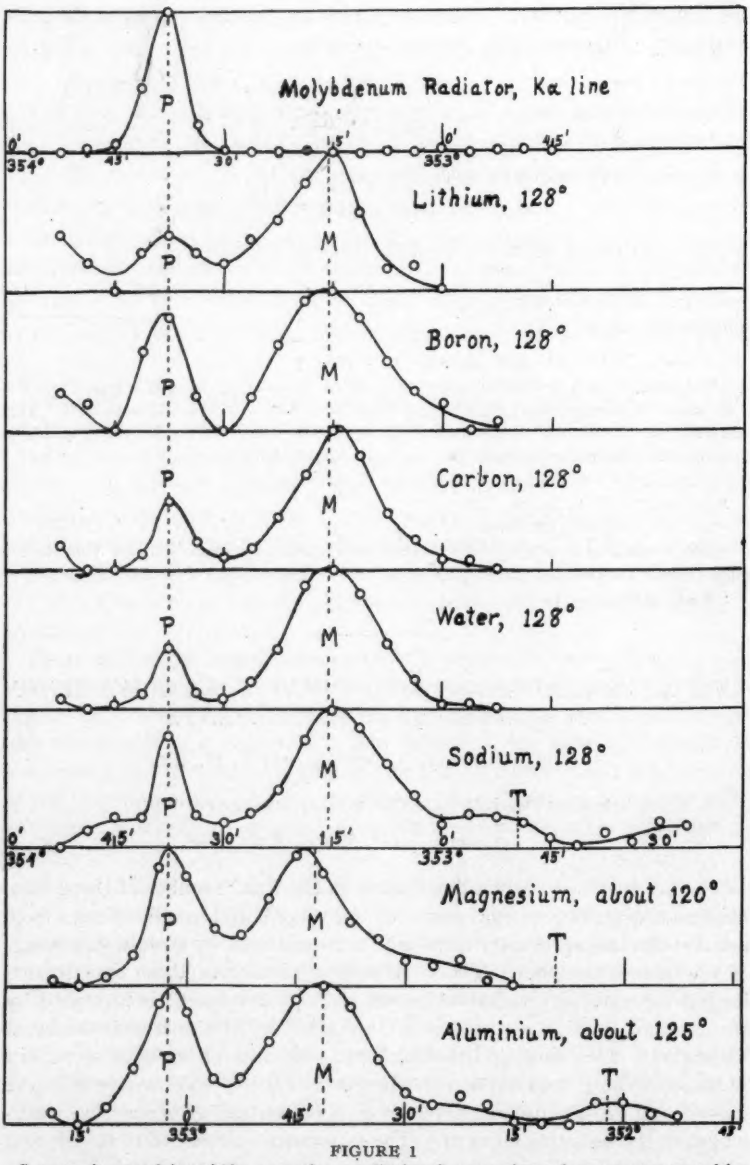

FIGURE 1

Spectra from calcite of the secondary radiation from various elements, traversed by X-rays from a molybdenum target. P marks the position of the primary K_α peak, M the theoretical position of the modified peak, and T the peak of the "tertiary radiation" according to Clark, Stifler and Duane's experiments.

by one of the writers for measuring the change of wave-length of X-rays scattered by carbon.[1] In the present work, however, instead of a pair of slits to direct the beam falling on the crystal, a collimator was employed composed of a pile of sheets of lead foil separated by strips of lead foil.[2] This device, due to W. Soller, enabled us to secure greater intensity than in the earlier measurements. The water cooled X-ray tube was operated at about 65 kilovolts peak and 40 milliamperes.

The samples of lithium, carbon and sodium were in the form of cylinders of 8 mm. diameter, and the boron and water were held in a very thin walled, waxed paper cylinder of the same diameter. In order to secure greater intensity, the samples of magnesium and aluminium were in the form of flat plates. These radiators were clamped in turn approximately 2.5 cm. from the focal spot of the X-ray tube, in such a position that they would scatter the rays into the collimating slits at about 125°. A slight surface oxidation of the lithium and sodium occurred during the course of the experiments. The boron used was in the amorphous form, which contains 4 or 5 per cent of oxygen. Our sample also had traces of silica, and the paper container introduced small amounts of carbon and hydrogen. The rays scattered from the water may be considered characteristic of oxygen, with hydrogen and some carbon from the paper container as impurities. The carbon (graphite), magnesium and aluminium presumably had impurities only in relatively small amounts.

The results of our measurements with radiators of molybdenum, lithium, boron, carbon, oxygen (water), sodium, magnesium and aluminium are shown in figure 1. The spectra from the first six radiators were taken without changing the adjustments. For the last two elements, the width of the slits of the diaphragm was increased from 0.1 to 0.2 mm. in order to secure greater intensity. The readjustment resulted in an altered zero point, as is indicated in the figure.

The important point in this figure is that the spectra obtained from the various elements are almost identical in character. In every case an unmodified line P occurs at the same position as the fluorescent Mo Kα line, and there is a modified line whose peak is within experimental error at the position M, calculated from the quantum change of wave-length formula given above. There is also perhaps some evidence in the cases of sodium and aluminium for a hump at the position T, where according to the experiments of Clark, Stifler and Duane the peak of the line due to "tertiary radiation" should appear.

In view of the consistency of the results for the different elements, we feel that these experiments show beyond question the reality of the spectrum shift predicted by the quantum theory of scattering.

[1] A. H. Compton, *Physic. Rev.*, **22**, 409 (1923).

[2] *Cf.* W. Soller, *Ibid.*, **23**, 292 (1924).

QUANTIZATION IN SPACE AND THE RELATIVE INTENSITIES OF THE COMPONENTS OF INFRA-RED ABSORPTION BANDS

By Edwin C. Kemble

Jefferson Physical Laboratory, Harvard University

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The purpose of this article is to show that the application of the Bohr correspondence principle to the theoretical evaluation of the relative intensities of the lines in the infra-red absorption bands of diatomic gases leads, in the case of HCl band at 3.5μ, to a definite conflict with the experimental facts if the molecule is treated in the usual way as a degenerate system with one quantum condition for two rotational degrees of freedom. On the other hand, if we assume a slight precession of the orbital plane about the lines of the earth's magnetic field, thus introducing space quantization of the orbital planes, the difficulty is removed. The discussion is based on the interpretation of the HCl band in terms of half quantum numbers first suggested by Kratzer[1] and recently confirmed for HCl by Colby's study of the lines which appear in the neighborhood of 3.5μ when the gas is heated.[2]

Let n and m denote the usual vibrational and rotational quantum numbers. Treating the system as degenerate, let us consider the absorption line due to jumps from the state n'', m'' to the state n', m'. (Primes will denote the state of greater energy.) Let α_ν be the absorption coefficient for the frequency ν and let

$$\alpha_{n',m'}^{n'',m''} \equiv \int \alpha_\nu d\nu, \tag{1}$$

where the integral is carried over the absorption line in question. The integral absorption coefficient thus defined is related to the number of molecules in the initial state $N_{n'',m''}$ and to the Einstein probability coefficient for this jump, $B_{n',m'}^{n'',m''}$ by the formula

$$\alpha_{n',m'}^{n'',m''} = \nu N_{n'',m''} B_{n',m'}^{n'',m''} \times \text{Constant}. \tag{2}$$

Introducing the usual statistical formula for $N_{n'',m''}$ in terms of the *a priori* probability $p_{n'',m''}$ and the energy $W(n'',m'')$ of the initial state, we obtain

$$\alpha_{n',m'}^{n'',m''} = C\nu p_{n'',m''} e^{-W(n'',m'')/kT} B_{n',m'}^{n'',m''}. \tag{3}$$

Einstein[3] has shown that $B_{n',m'}^{n'',m''}$ is related to the probability coefficient for emission by spontaneous transition from n',m' to n'',m'' by the formula

$$B_{n',m'}^{n'',m''} = \frac{c^3}{8\pi h\nu^3} \frac{p_{n',m'}}{p_{n'',m''}} A_{n'',m''}^{n',m'} \tag{4}$$

Hence (3) becomes

$$\alpha_{n',m'}^{n'',m''} = p_{n',m'} e^{-W(n'',m'')/kT} A_{n'',m''\nu}^{n',m'} \nu^{-2} \text{ Const.} \quad (5)$$

$A_{n'',m''}^{n',m'}$ may be evaluated by Bohr's correspondence principle in terms of the coefficients in the Fourier expansion of the electric moment M of the molecule. Let $X(n,m,\Delta n,\Delta m)$ denote the amplitude of the harmonic component of M_x correlated with the jump in question and evaluated for an orbit with quantum numbers n and m.[4] Let Y and Z denote the corresponding amplitudes in the Fourier expansions of M_y and M_z, respectively. Then according to Bohr and Kramers[5]

$$A_{n'',m''}^{n',m'} = \nu^3 \overline{(X^2 + Y^2 + Z^2)} \times \text{Const.} \quad (6)$$

where the sum of the squares of the amplitudes is to be averaged over the hypothetical classical orbits between the initial and final states. The method of averaging is uncertain and in fact it is uncertain whether any definite scheme of averaging will always give rigorous results. We assume, however, that as a first approximation we may write

$$\overline{X^2} = [X(\bar{n},\bar{m},\Delta n,\Delta m)]^2 \quad (7)$$

with similar equations for $\overline{Y^2}$ and $\overline{Z^2}$. $\bar{n}$ and $\bar{m}$ denote the arithmetic means of the values of n and m for the initial and final orbits. Equation (5) now becomes

$$\alpha_{n',m'}^{n'',m''} = \text{Const.} \times \nu p_{n',m'} e^{-W(n'',m'')/kT} [X^2(\bar{n},\bar{m}) + \overline{Y^2}(\bar{n},\bar{m}) + Z^2(\bar{n},\bar{m})] \quad (8)$$

The electric polarization M is assumed to be proportional to the vector distance between the atomic nuclei. Neglecting for the present any possible precession about lines of magnetic force and assuming a non-gyroscopic molecular model, the motion of the terminal point of M must be uniplanar and compounded of a vibration of frequency ω_v and a rotation of average angular velocity $2\pi\omega_r$. It is convenient to introduce two sets of axes x,y,z and x',y',z' with a common origin. The first set is fixed in space, the xy plane being the plane of the orbit. The z' axis is coincident with z but the x' and y' axes rotate with constant angular velocity $2\pi\omega_r$ about the z axis. The orbit in the $x'y'$ plane is evidently periodic. If the molecular rotation took place at a uniform rate it would degenerate into a straight radial line, but owing to the variation in angular velocity with the separation of the atoms it is actually of an elongated oval form with major axis radial. The Fourier development is conveniently written in the form

$$M_{x'} + iM_{y'} = \sum_{\tau=0}^{\tau=\infty} \left\{ R_\tau e^{2\pi i(\tau\omega_v t + \delta)} + R_\tau' e^{-2\pi i(\tau\omega_v t - \delta)} \right\} \quad (9)$$

To avoid undue complication the discussion is at this point limited to

the so-called "fundamental" vibration-rotation band which is correlated with the fundamental vibration frequency of the Fourier analysis. Unity is then the only value of Δn and τ to be considered and equation (9) can be replaced by

$$M_{x'} + iM_{y'} = Re^{2\pi i(\omega_v t + \delta)} + R'e^{-2\pi i(\omega_v t - \delta)} \tag{10}$$

Shifting to fixed axes is equivalent to multiplying by $e^{2\pi i\omega_r t}$. Hence

$$M_x + iM_y = Re^{2\pi i[(\omega_v + \omega_r)t + \delta]} + R'e^{-2\pi i[(\omega_v - \omega_r)t - \delta]} \tag{11}$$

If we treat the orbit as a radial line in order to get a first approximation, R and R' are equal, but an analysis of the ratio of the major and minor axes required by the law of the conservation of angular momentum shows that as a second approximation we may write

$$R = A\left(\frac{\omega_v}{\omega_v + \omega_r}\right)^2; \quad R' = A\left(\frac{\omega_v}{\omega_v - \omega_r}\right)^2. \tag{12}$$

A and ω_v vary slightly with the rotational quantum number of the orbit for which they are evaluated, but this variation will be neglected in comparison with that of ω_r. R and R' are the amplitude coefficients for M_x and M_y corresponding to the values $\Delta m = +1$ and $\Delta m = -1$ respectively. As the frequency absorbed is in the first case $\omega_v + \omega_r$ and in the second $\omega_v - \omega_r$, we have

$$X = Y = \text{Const.}/\nu^2; \qquad Z = 0. \tag{13}$$

Equation (8) now becomes

$$\alpha_{1,m'}^{0,m''} = \frac{\text{Const.}}{\nu^3} \times p_{n',m'}e^{-W(0,m'')/kT} \tag{14}$$

The above equation is in sharp disagreement with the experimental facts. The two central components of the HCl band at 3.5μ are of approximately equal intensity,[6] though according to (14) the high frequency component ($m' = 3/2$) should be much more intense than the low frequency component ($m' = 1/2$). The exponential factor and the factor $1/\nu^3$ are not very different in the two cases so that the intensity ratio is nearly equal to the ratio of the *a priori* probabilities of the states having the rotational quantum numbers 3/2 and 1/2, which is two or three to one according to one's taste in *a priori* probability formulas.

The trouble with the theory lies in the factor $p_{m'n'}$ which depends on the value of m for the final state of the absorption process In order to obtain even approximate agreement with the experimental facts we must replace this factor by one depending in a more or less symmetrical way on the initial and final values of m. The desired result is effected if we assume

the existence of a slight precessional motion about the lines of an external magnetic field and so pass to the discussion of a non-degenerate motion in which the number of quantum conditions is equal to the number of degrees of freedom and in which the *a priori* probabilities of all stationary states are the same.[7] In analogy with the theory of the Zeeman effect the angular momentum parallel to the field is assumed to be $rh/2\pi$, where r is a quantum number associated with the precessional motion. It is further assumed (Hypothesis 1) that for any given m, r takes on the values $\pm 1/2,\ \pm 3/2, \ldots \pm m$.[8] An alternative possibility[9] (Hypothesis 2) to be considered later gives r the values $0,\ \pm 1,\ \pm 2, \ldots \pm (m - 1/2)$. The molecule being presumably diamagnetic, the energy values will be sensibly independent of r and the emission frequencies for jumps with the same m' and m'' but different values of r' and r'', will be the same.

To get the new Fourier development we may assume that the x,y,z system of axes previously considered is not really stationary, but rotates with a small constant angular velocity $2\pi\sigma$ about the lines of magnetic force. We choose a new set of fixed axes ξ, η, ζ such that ζ coincides with the field. Let θ denote the constant angle between z and ζ. Without loss of generality we may assume that the x-axis lies in the ξ-η plane and makes an angle $2\pi\sigma t$ with the ξ axis. Shifting from the precessing to the fixed axes, we use the transformation equations

$$M_\xi + iM_\eta = \left\{\left(\frac{1+\cos\theta}{2}\right)(M_x + iM_y) + \left(\frac{1-\cos\theta}{2}\right)(M_x - iM_y)\right\} e^{2\pi i\sigma t};$$
$$M_\zeta = M_y \sin\theta. \tag{15}$$

Hence

$$M_\xi + iM_\eta = \left(\frac{1+\cos\theta}{2}\right)\left\{Re^{2\pi i\{(\omega_v + \omega_r + \sigma)t + \delta\}} + R'e^{-2\pi i\{(\omega_v - \omega_r - \sigma)t - \delta\}}\right\}$$
$$+ \left(\frac{1-\cos\theta}{2}\right)\left\{Re^{-2\pi i\{(\omega_v + \omega_r - \sigma)t + \delta\}} + R'e^{2\pi i\{(\omega_v - \omega_r + \sigma)t - \delta\}}\right\}$$
$$M_\zeta = \sin\theta\{R\sin\{2\pi(\omega_v + \omega_r)t + \delta\} - R'\sin\{2\pi(\omega_v - \omega_r)t - \delta\}. \tag{16}$$

The coefficients ± 1 of ω_r and σ in the development give the values of Δm and Δr for the jumps with which these terms are associated. Let $\Xi_{\Delta r}^{\Delta m}(\bar{m},\bar{r})$, $H_{\Delta r}^{\Delta m}(\bar{m},\bar{r})$, $Z_{\Delta r}^{\Delta m}(\bar{m},\bar{r})$ be the ξ, η, ζ components of the amplitudes evaluated for the mean orbit $\bar{m}$, $\bar{r}$. Then since $\cos\theta = r/m$

$$\begin{aligned}
&\Xi_{-1}^{+1}(\bar{m},\bar{r}) = \tfrac{1}{2}R(1 + \bar{r}/\bar{m});\quad \Xi_{-1}^{+1}(\bar{m},\bar{r}) = \tfrac{1}{2}R(1 - \bar{r}/\bar{m});\\
&\Xi_{+1}^{-1}(\bar{m},\bar{r}) = \tfrac{1}{2}R'(1 - \bar{r}/\bar{m});\quad \Xi_{-1}^{-1}(\bar{m},\bar{r}) = \tfrac{1}{2}R'(1 + \bar{r}/\bar{m});\\
&H_{\Delta r}^{\Delta m}(\bar{m},\bar{r}) = \Xi_{\Delta r}^{\Delta m}(\bar{m},\bar{r});\\
&Z_{0}^{+1}(\bar{m},\bar{r}) = R\sin\bar{\theta} = R\sqrt{1-(\bar{r}/\bar{m})^2};\quad Z_{0}^{-1}(\bar{m},\bar{r}) = R'\sqrt{1-(\bar{r}/\bar{m})^2}.
\end{aligned} \tag{17}$$

All other coefficients vanish.

Equation (8) retains its form except that $p_{n',m'}$ drops out and the squares of the amplitudes are to be summed up over all values of $\bar{r}$ and Δr consistent with the values of $\bar{m}$ and Δm under consideration. As the discussion is restricted to a definite pair of values of n' and n'' and as ± 1 are the only values of Δm to be considered, the notation for the absorption intensities can be simplified. Let $\alpha^{+1}(\bar{m})$ denote the integral absorption coefficient for the component of the high frequency branch ($\Delta m = +1$) having the mean rotational quantum number $\bar{m}$. Let $\alpha^{-1}(\bar{m})$ denote the coefficient for the corresponding line of the negative branch. Then

$$\alpha^{\pm 1}(\bar{m}) = \text{Const.}\ \frac{1}{\nu^3} S(\bar{m}) e^{\frac{-W(\bar{m} \mp 1/2)}{kT}}, \tag{18}$$

where, neglecting a constant factor, $\nu^4 S(\bar{m})$ represents the sum of the squares of the amplitude coefficients. From equations (17) and (12) we introduce the definition

$$S(\bar{m}) = \tfrac{1}{2} \sum_{\bar{r}} \left\{ \tfrac{1}{2}(1 + \bar{r}/\bar{m})^2 + \tfrac{1}{2}(1 - \bar{r}/\bar{m})^2 + \left((1 - \left[\frac{\bar{r}}{\bar{m}} \right]^2 \right) \right\} \tag{19}$$

The terms $\tfrac{1}{2}\left(1 + \frac{\bar{r}}{\bar{m}}\right)^2$ and $\tfrac{1}{2}\left(1 - \frac{\bar{r}}{\bar{m}}\right)^2$ are to be summed from $\bar{r} = -\bar{m}$, to $\bar{r} = +\bar{m}$, while the term $1 - \left(\frac{\bar{r}}{\bar{m}}\right)^2$ is to be summed from $-(\bar{m} - \tfrac{1}{2})$ to $+(\bar{m} - \tfrac{1}{2})$. Equation (19) is readily simplified and yields

$$\begin{aligned} S(\bar{m}) &= \tfrac{1}{2} + \sum_{\tau=1}^{\tau=\bar{m}} \left\{ 1 + \left(\frac{\tau}{\bar{m}}\right)^2 + 1 - \left(\frac{\tau - \tfrac{1}{2}}{\bar{m}}\right)^2 \right\} \\ &= 2\bar{m} + 1 - 1/(4\,\bar{m}). \end{aligned} \tag{20}$$

$S(\bar{m})$ differs only by the small term $1/(4\,\bar{m})$ from the mean of the *a priori* probabilities of the initial and final groups of states involved in the absorption line under consideration. It is possible that some other method of averaging the squares of the amplitudes would eliminate this term. Equations (18) and (20) reduce the theoretical difference in the intensities of the central components of the HCl band under discussion from 320% to about 7%, and are in qualitative agreement with the observations on the outer components.

The alternative hypothesis 2 regarding the possible values of r leads to the formula

$$S(\bar{m}) = 2\bar{m} - 1/(4\,\bar{m}) \tag{21}$$

which seems not to fit the data as well as (20). A more detailed discussion of this point will be given in a later paper.

[1] A. Kratzer, *Ber. d. Bay. Akad.*, p. 107, 1922.

[2] W. Colby, *Astroph. J., Chicago*, **58**, p. 303, 1923.

[3] A. Einstein, *Physik. Zs.*, **18**, 121, 1917.

[4] Each of the quantum numbers is defined as $1/h$ times the corresponding phase integral. The quantum numbers of the hypothetical non-quantized orbits between the initial and final states are fractional.

[5] H. A. Kramers, *Det Kgl. Danske Vidensk. Selsk. Skrifter*, 8 Raekke, **3**, 1919, p. 330.

[6] Cf. E. S. Imes, *Astroph. J.*, **50**, p. 260, 1919; Brinsmade and Kemble, *Proc. Nat. Sci.*, **3**, p. 420, 1917.

[7] Cf. Bohr, Quantum Theory of Line-Spectra, I, p. 26, *Det Kgl. Danske Vidensk. Sesk. Skrifter*, 8. Raekke, IV.

[8] A. Landé, *Zs. Physic.*, **11**, 357 (1922); R. C. Tolman, *Physic. Rev.*, 2nd Ser., **22**, p. 470, 1923.

[9] A. Landé, *Physik. Zs.*, **24**, p. 441, 1923.

THE BALLISTIC (AIR RESISTANCE) FUNCTION

BY L. THOMPSON

NAVAL PROVING GROUND, DAHLGREN, VA.

Communicated, April 10, 1924

A projectile in flight is subject to retarding forces of which the resultant may attain a value greatly exceeding the weight, particularly in the case of small calibers. It is customary to express the retardation as a function of the velocity, and the determination of this function constitutes one of principal experimental problems of exterior ballistics. It appears in the equations of motion

$$d(v \cos \Theta) = -f(v)/C \cos \Theta \, dt, \qquad d(v \sin \Theta) = -f(v)/C \sin \Theta \, dt - g \, dt$$

involving the quantity $C = m\delta_0/\pi r^2 i\delta$, a measure of the ballistic or "carrying" capacity of the projectile. In addition to the relative air density, mass and cross section normal to the direction of flight, C includes a factor i frequently assumed to be completely defined by the shape.

The variation of the air resistance $R = \pi r^2\delta/\delta_0 . i f(v)$ with changes in velocity can be represented advantageously by means of the coefficient $R/v^2r^2\delta = \pi i K(v)/\delta_0$, of which $v^2K(v) = f(v)$.

If a particular function, $K(v)$, is made the basis for the comparison of projectiles of different shapes and magnitudes, the factor i, pertaining to individuals, is not a constant. This is otherwise signified by the statements $K_1(v) = i_1(v)K(v)$, $K_2(v) = i_2(v)K(v)$, ... where the subscripts refer to corresponding projectiles.

For velocities in excess of the velocity of sound, a rapid dissipation of energy occurs through the production of air waves. These are well shown in spark photographs of the motion. Among the best are those recently taken at the Case School of Applied Science by Professor D. C. Miller

and Mr. P. P. Quayle. Three from that laboratory are reproduced in figure 1.[1] The second term of the Sommerfeld expression[2]

$$f(v) = av^2 + A(1-v_s^2/v^2)$$

is intended to account for the effect of wave resistance and was suggested

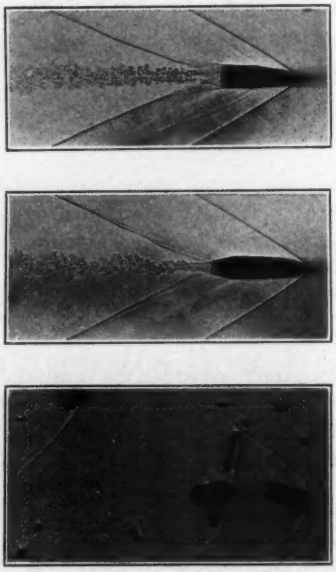

FIGURE 1

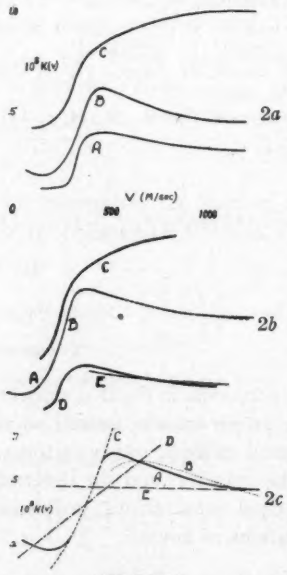

FIGURE 2

2a.

A Krupp normal 10 cm. shell
B Charbonnier
C Krupp cylinder (10 cm.)

2b.

A Charbonnier
B Sommerfeld
C Krupp cylindrical bullet
D Krupp Infantry S bullet
E .30 caliber experiments

2c.

B Chapel-Vallier $K(v) = (a + bv)v^{-2}$
C Piton-Bressant b_1v^2
D Didion $a_1 + b_2v$
E Newton, Helie c

by the analogy of an electromagnetic field in which an electron was assumed moving with a velocity greater than the velocity of light.

In figure 2*a* are several $K(v)$ curves, *A* representing the Krupp normal 10 cm. shell and *C* the Krupp cylinder, the form factor being 1 in either case.[3] The Charbonnier (*B*) curve presents the Gâvre data.[4] These are not so definitely related to a single projectile. The constants of the Sommerfeld form were determined with reference to this curve, and the dotted line of figure 2*b* shows that it provides as excellent description of the experimental results for velocities above that of sound. This is more

suitable than most of the proposed empirical relations, some of which are indicated in figure 2c. Curve B refers to the linear resistance law, suggested by Chapel and since used by Vallier and others.[5] It is fairly satisfactory over a considerable range of high velocities.

If all significant variables are included, the resistance may be assumed of the form

$$R = v^2L^2\delta.F(vL\delta/\mu, v/v_s) \qquad (for\ v > v_s)$$

where L indicates the magnitude of the projectile, μ the coefficient of viscosity of air, and v_s the velocity of sound. Use of data obtained with one projectile for the prediction of retardation in the case of another, geometrically similar, requires a selection of circumstances such that F will be unchanged. But it is impossible to maintain both $vL\delta/\mu$ and v/v_s for even small variations of L (except through a change in the properties of the medium). The ratio for two projectiles at any velocity must therefore be written

$$R_1/R_2 = L_1^2F(vL_1\delta/\mu, v/v_s)/L_2^2F(vL_2\delta/\mu, v/v_s) = i_1L_1^2/i_2L_2^2$$

and $i_2 = i_1\psi(L, v)$ is undetermined until experiments have been conducted with the second projectile. There is no evidence that $\partial\psi/\partial L$ has a zero value. Data for similar shells of different calibers show that neither the resistance per unit area at one velocity nor the ratio of resistances per unit area at two velocities is maintained for a variation of the diameter.[6] That i is a function of the velocity is fairly well established. Von Eberhard has expressed the Krupp experience in the formula[7]

$$i_r^{-1} = a_r + b_rv + c_rv^{-1}$$

comparing several forms with the Krupp normal shell.

The third figure refers to the resistance coefficient defined by the two base variables, and suggests a statement of the problem. It appears that the surface $z = F(x, y)$ should be identified for each type of projectile. Experiments with a particular projectile (L_r), establish a section, the base of which is the diagonal line $x/y = v_sL_r\delta/\mu$, a constant. Any section constitutes a two dimensional representation of the form

$$R/v^2L^2\delta = F_1(v) = \pi K(v)/4\delta_0$$

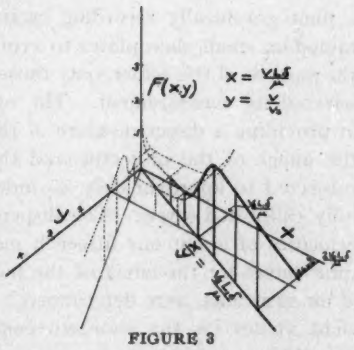

FIGURE 3

The heavy line was drawn from data for the Krupp normal shell. It may be noted that for $v < v_s$, $\delta F_x = 0$ and the surface in this region is cylindrical.

The requirement for dynamical similarity is reduced to that of the maintenance of one quantity, $vL\delta/\mu$, and a moderate range of magnitudes may be referred to experiments with a single projectile.

Data from the experiments of Bashforth, Mayewski, Hojel and Krupp, arranged to best advantage by General Siacci,[8] were used for practically all ballistic tables before the period of the war. Lack of a sufficiently precise chronograph was responsible for large uncertainties in the results. The values adopted were obtained by an averaging process, associating data from many projectiles fired under conditions which were almost certainly not comparable. The recent Krupp experiments and those at the Gâvre Proving Ground have produced more significant observations, the former having the advantage of association with shells of definite size.

The mean deflection of the principal axis of a projectile with respect to the tangent to the trajectory may change considerably during a long flight. Since the air resistance is seriously affected by such a change, it is desirable that experiments should be made with an apparatus of sufficient precision to avoid the necessity for long distances and time intervals. In the computation of trajectories it is, of course, required to know the variation in resistance for small and measured changes in this angle.

An optical chronograph,[9] adapting a beam of light as a screen, has been used to develop a technique for the measurement of air resistance which appears to meet most of the requirements. With .30 caliber aluminum infantry projectiles of the form shown in figure 1*a*, individual resistance values have been secured during a flight of from three to four meters, which fall acceptably close to a comparable expression of the ballistic function (table 1 and figure 4).

The weights of the bullets were of the order of 2.7 grams. The distances between crossings of the beam of light, about 70 cm., were measured with a photographically recording carriage.[9] The velocity records were obtained on small glass plates to avoid the effect of film shrinkage, and, for the purpose of this paper, only those which showed no observable yaw were selected for measurement. The optical chronograph has an advantage in providing a direct measure of the axial dsiplacement, in the length of the image of the projectile and the slope of its base. Each plate was subjected to approximately 35 independent determinations, using a carefully calibrated screw. The dispersion indicates a probable error in the velocities of about one fifteenth meter-second. Small corrections in the time values for the effect of the holding magnet of the chronograph and of air resistance were determined.[9] Observations were reduced to equivalent values for the standard conditions of temperature and pressure, $T = 16°$, $p = 760$ mm.

The resistance was calculated from the expression for the time mean $R = \Delta mv/\Delta t$ and associated with the mean velocity. It was established,

through an examination of the series which represent respectively the resistance, velocity, and distance as functions of the time that (for the short intervals of these experiments) the error in associating the means

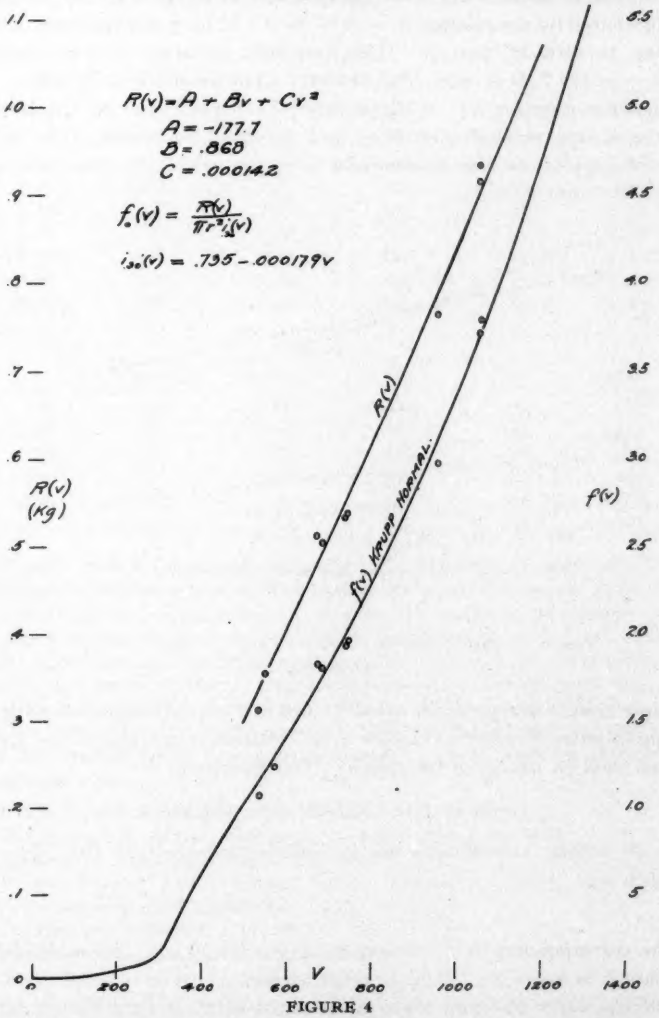

FIGURE 4

was well within the limits of experimental error. For the majority of cases it will be necessary to use a series defining these quantities at certain positions or instants. Professor F. R. Moulton has recently developed a

set of difference equations particularly adapted for the computation of velocity, resistance, and rate of change of resistance at four or more equidistant screens.[10]

The data of table 1 are shown graphically in figure 4. The resistance is represented by the relation $R = A + Bv + Cv^2$ for velocities from 500 M. per sec. to 1100 M. per sec. The constants, adjusted by least squares, are $A = -177.7$, $B = .868$, $C = .000142$. The force is in kilograms.

The form function $i(v) = R(v)\delta_0/\pi\delta r^2 v^2 K(v)$ was found on the basis of (a) the Krupp normal shell $K(v)$ and (b) the Charbonnier $K(v)$ which therefore applies to the Sommerfeld $K(v)$ (for which, in this case, $a = 3.9 \times 10^{-6}$ and $A = 1$).

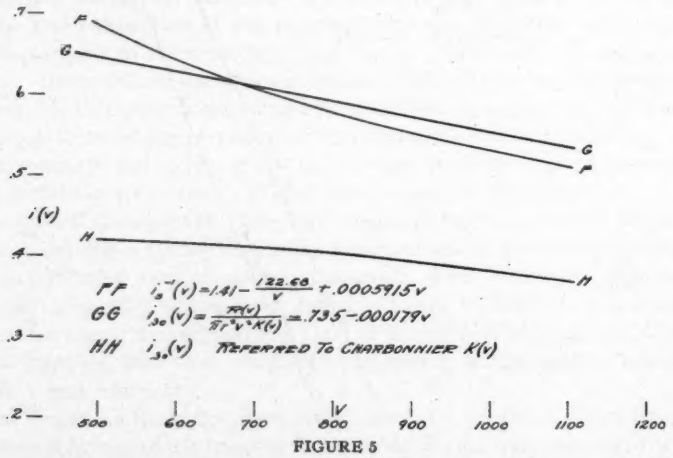

FIGURE 5

These results are given in table 2, and in figure 5 compared with the Krupp infantry S bullet $i(v)$ which is also expressed in terms of the Krupp normal shell by means of the relation (von Eberhard)

$$i_s^{-1}(v) = 1.41 - 122.68v^{-1} + .0005915\,v$$

The .30 caliber experiments are accurately described by the linear expression

$$i_{.30}(v) = .735 - .000179\,v$$

The corresponding $K(v)$ curves, $K_s(v) = i_s(v)K(v)$, $K_{.30}(v) = i_{.30}(v)K(v)$ are shown in figure 2b, D and E, respectively.

The $f(v)$ curve of figure 4 was constructed with the data for the Krupp normal shell, and the .30 caliber observed values, $f_0(v)$ (circles), were obtained from the observed resistances, $R_0(v)$, and the .30 caliber form function $f_0(v) = R_0(v)/\pi r^2 i_{.30}(v)$.

These experiments were carried out in the physical laboratory of Kalamazoo College under a National Research Fellowship. Mr. Nils Riffolt constructed the apparatus and contributed extensively to the development. A grant was received from the American Association for the Advancement of Science.

TABLE I

M / s	M/SEC. / v	Δv	KGM. / $r_0(v)$	$f_0(v)$	$10^3 k_0(v)$
3.11	541	6.44	.310	1.07	2.33
3.11	577	6.91	.354	1.23	2.33
2.77	679	7.30	.511	1.83	2.43
2.77	691	7.33	.504	1.80	2.32
3.46	745	8.51	.532	1.93	2.10
2.77	754	7.12	.536	1.95	2.07
3.11	965	8.63	.762	2.97	1.80
3.46	1060	10.34	.914	3.71	1.78
2.77	1061	8.44	.933	3.78	1.82

TABLE II

v	$R(v)$	$10^3 k_{.30}(v)$	$i\,k_{.30}(v)$	$i^{k}_{3}(v)$	$i^{ch}_{.30}(v)$
500	.293	2.57	.642	.685	.420
600	.394	2.40	.625	.641	.416
700	.501	2.24	.615	.606	.412
800	.609	2.09	.596	.578	.405
900	.720	1.95	.573	.554	.397
1000	.835	1.83	.551	.532	.381

[1] Some of these have appeared in a paper in *J. Franklin Inst.*, (May 1922, V. 193, P. P. Quayle). Professor Miller kindly supplied the prints shown in the figure, which were taken from the original negatives. 1*a* shows the normal shape but with a small hole drilled through the body of the projectile (perpendicular to the axis). This accounts for the secondary disturbance near the center. 1*b* was made with a "boat tail" projectile, the purpose of which is to decrease the dissipation of energy at the base. The area of the turbulent wake is noticeably smaller. 1*c* is suggestive of the increase of resistance in the case of a deflected projectile. The two large pieces are a "dum-dum" bullet and detached jacket, (after passing through a board). They are followed by a wake of large area.

[2] F. Klein and A. Sommerfeld, *Theorie des Kreisels*, **4**, Section C.

[3] C. Cranz and K. Becker, *Handbook of Ballistics*, 1921, **1**, pp. 51–2.

[4] Charbonnier, *Traité de Balistique Extérieure*, **1**, 1921.

[5] W. von Scheve, "Law of Chapel-Vallier," *Kriegstechn. Zeitsch.*, 1907. Vallier, *Balistique Extérieure*, (2nd Edition).

[6] C. Cranz and K. Becker, **1**, pp. 36–7.

[7] Von Eberhard, *Artill. Monatshefte*, Sept. 1912.

[8] Siacci, *Rivista artigl. gen.*, **1**, 1896.

[9] L. Thompson, "The Motion of a Falling Chronograph Projectile," *Proc. Nat. Acad. Sci.*, **9**, No. 9.

[10] Not yet published.

RUPPIA AND ITS ENVIRONMENTAL FACTORS

BY WILLIAM ALBERT SETCHELL

DEPARTMENT OF BOTANY, UNIVERSITY OF CALIFORNIA

Communicated, June 12, 1924

Ruppia maritima L. is a widespread and variable species growing in shallow brackish water over most of the northern and to some extent, at least, even over the southern hemisphere. Taxonomists vary in recognizing one or two species with a numerous and rather confusing category of subspecies, proles, varieties and forms. The North American variations have been most clearly treated of by Fernald and Wiegand in their paper in *Rhodora* of July, 1914. The presence of Ruppia in a number of localities about Richardsons Bay, running north from San Francisco Bay into Marin County, California, provided an accessible field for observation and experiment. The writer had the coöperation of Miss Mildred Bennett, graduate student in botany, and also of Mr. Harold E. Parks, Technician in the Department of Botany, in carrying out some preliminary investigations under natural conditions and under more or less controlled conditions in the laboratories. His obligations to his two co-workers are very considerable.

Some eleven stations within a distance of a mile or thereabouts apart and representing various different conditions were visited regularly, observations made, and seeds or cuttings brought into the laboratory for observation under control. Stations 1 and 2 were situated directly on the shore of Richardsons Bay and were subjected to tidal conditions of being alternately left exposed to the air or covered with shallow or even deep water. The Ruppias at these stations were perennial plants, persisting throughout the year, but actively vegetating, flowering and fruiting only in the summer and autumn months. The plants of these stations bore long, spirally coiled peduncles and the majority of their fruits were robust, rounded without pronounced beak, and only slightly curved, but a few were slender, flattened and strongly curved. They seemed to belong to the var. *longipes* Hagström of Fernald and Wiegand's account (p. 125). At the other stations (viz., 3–11), shallow pools or ponds, connected with the outer water by ditches or by overflow at highest tide, the Ruppia plants were ephemeral, dying down in late winter and reappearing in early spring. They may be said to be annual. They bore short peduncles, recurved at anthesis and in fruit, and they bore fruits mostly narrow, flattened, strongly curved and with a pronounced beak but also, sparingly, they bore in the same flowers with the first sort, stout fruits of the type most prevalent in the long peduncled forms at stations 1 and 2. They seem to correspond to *Ruppia maritima* var. *rostrata* Agardh of Fernald

and Wiegand's account (p. 125). Fairly frequent observations were made at station 2, stations 4 and 7, and station 10. The last station dries up in late spring and is not filled with water again until the winter rains.

Plants brought in were grown in large glass beakers of 2 liters or more capacity, in a tower room, lighted from four sides, provided with electric heaters and controlled by a thermostat to keep the temperature within fixed limits. The following points made themselves manifest in the preliminary studies:

1. The *longipes*-form was distinctly perennial, quiescent during the cooler months but forming mats on the muddy shores of the tidal flats. In the tower room, in cultures, it grew all the year, bloomed fairly freely, but matured little fruit.

2. The *rostrata*-form disappeared utterly during the cooler months and even died out in some of the hotter months entirely or nearly so. In the tower room, in cultures, it flourished throughout the year and blossomed and fruited abundantly.

3. The *longipes*-form varied considerably as to length of peduncle, was always comparatively long, erect, at least with a tendency thereto, and more or less coiled, especially after anthesis.

4. The *rostrata*-form varied little in the length of the peduncle, which was always short, only a few centimeters in length, and recurved, more pronouncedly after anthesis. The length and behavior was the same near the surface or submerged some considerable distance.

5. The forms of *Ruppia* investigated presented two types of fruits, viz.; (1) a robust type, only slightly one-sided and little or not at all compressed, and (2) a flattened curved type. Each of these types of fruit has been considered characteristic of a particular variety or form and the differences used to separate the various forms and varieties of *Ruppia maritima* into two groups (cf. Fernald and Wiegand).

6. The robust, symmetrical type of fruit is commonly found in the same flower with the compressed curved type but is more numerous proportionally in the *longipes*-form than in the *rostrata*-form, where the compressed curved type predominates.

7. Both forms have long podogynes, but even in the same flower, the podogynes of the robust type of fruit are much longer than those of the compressed type of fruit.

8. In their native habitats, the salinity of the water varied from 0.9162 to 4.5872 grams of salt to 100 cc. of solution, the *longipes*-form in solutions of 2.9560 to 100, while the *rostrata*-form occurred in solutions from 0.9162 to 2.4308 to 100. In cultures of tap water, both forms flourished in solution of 0.0683 grams to the 100 cc. of solution.

9. The pH values of the waters naturally inhabited by the plants regularly increased as the season of growth and reproduction advanced from

pH 7.4 or pH 7.6 to pH 9.6. In cultures no such regular increase of pH value was noted but the pH remained around the neutral point with a slight leaning towards the basic side of the scale, but not going higher than pH 8.4. There was a rise in pH values in culture from morning to afternoon.

10. The daylight-darkness ratio seemed definite for the plants in the field, but those in cultures grew and reproduced continuously throughout the year from one extreme of this ratio to the other.

11. Temperatures of the air and water were taken at the various stations at frequent intervals. On account of the shallowness of the water there was no general rule to be deduced, except that the progression from germination to anthesis and fruiting was carried through on a general rise of temperature and that cessation of activity accompanied the advent of cold weather. Shallow water, however, is quickly raised and as quickly lowered in temperature by insolation or the lack of it and the general changes of air temperature. Under controlled conditions, the temperature relations became manifest as given below.

12. In cultures in a tower room, lighted from all sides and kept at temperatures (by electrically controlled heat) varying from 18°C. to 35°, but with average of 22°C. to 24°C., both forms of Ruppia flourished and reached anthesis and fruited (*longipes*-form rarely, but *rostrata*-form regularly and uniformly).

13. It was observed that above 25°C., anthesis was slow and if long continued, ceased.

14 Germinations could be obtained but rarely at these higher temperatures. In the laboratories of the same building, however, and in cultures varying from 10° to 18°C., germinations were readily obtained but development beyond the seedling stage did not take place. Seedlings removed to the tower room (22 to 24°C.) proceeded to anthesis and fruiting.

15. From our observations and experiments, it appears that *Ruppia maritima* will flourish and carry through its life processes under a considerable range of salinity content, pH variation, and darkness-daylight ratios, the lim ts in any direction not having been reached under ordinary conditions. The temperature control, however, seems clearly indicated, viz., between 15 and 20°C. for germination and seedling development and 20°C. to 25°C. for vegetative growth and reproductive activity. The temperature relations agree in principle with those already observed for the varieties of *Zostera marina*, viz., 10°C. to 15°C. for germination and early stages of vegetation and 15°C. and 20°C. for reproduction.